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# Witten Deformation on noncompact manifolds and Landau-Ginzburg B-model

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For my family.

# Curriculum Vitæ

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## Education

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## Publications and Preprints

[1] Xianzhe Dai and Junrong Yan. *Witten Deformation for Noncompact Manifolds with Bounded Geometry*, Journal of the Institute of Mathematics of Jussieu, 1-38 (2021)

[2] Xianzhe Dai and Junrong Yan. *Witten Deformation on Non-compact Manifold: Heat Kernel Expansion and Local Index Theorem*, arXiv:2011.05468

[3] Xinxing Tang and Junrong Yan. *Calabi-Yau/Landau-Ginzburg Correspondence for Weil-Peterson Metrics and $tt^*$ Structures*, arXiv: 2205.05791

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## Abstract

Witten Deformation on noncompact manifolds and Landau-Ginzburg B-model

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Motivated by the Calabi-Yau/Landau-Ginzburg correspondence, we study BCOV-type torsion for Landau-Ginzburg B-models. To pave the way for understanding the Landau-Ginzburg model from the index theoretic point of view, we study Witten deformation, which was introduced in one of Witten's extremely influential papers in 1987, on non-compact manifolds. In Part I, we explore the analysis of Witten deformation on non-compact manifolds: we show the asymptotic growth of eigenvalues and the decay of eigenfunctions near infinity as well as the expansion and the estimate of the heat kernel for Schrödinger-type operators on noncompact manifolds. With heat kernel expansions of Schördinger-type operators, we define the Ray-Singer metric (analytic torsion) for the Witten deformation associated with some flat vector bundle and explore several nice properties of it (independence of metrics, Cheeger-Muller/Bismut-Zhang theorem, e.t.c.). Next, we move on to study Landau-Ginzburg B-models in Part II. In chapter 6, we investigate the genus-zero theory for Landau-Ginzburg B-models, and establish Calabi-Yau/Landau-Ginzburg correspondence for the $tt^*$ structure and the Weil-Peterson type metric.

# Contents

# Chapter 1

# Introduction

## 1.1 Background and Motivations

The mirror symmetry, first formulated in the 1990s, is a correspondence between the symplectic geometry (A-model) of a Calabi-Yau manifold $X$ and the complex geometry (B-model) of its mirror $X^\vee$. Since its formulation over three decades ago [1, 2], the mirror symmetry conjecture has had a profound influence on mathematics. The A-model on a Calabi-Yau manifold $X$ is the famous Gromov–Witten theory of $X$. The genus 0 theory of Calabi-Yau B-model is related to the variation of Hodge structure, whereas the higher genus B-model theory is known as BCOV theory [3], [4].

Simultaneously, physicists extended the above discussion to singularity theory, which became known as the Landau-Ginzburg model. A Landau-Ginzburg model is defined on the pair $(X, f)$, where $X$ is a complete noncompact Kähler manifold and $f$ is a holomorphic function. For the A-side, the Landau-Ginzburg A-model is constructed by Fan-Jarvis-Ruan [5] following Witten's proposal [6], which is known as FJRW-theory. Moreover, Chang-Li-Li [7] recently formulated FJRW-theory in an algebraic geometric formulation. On the B-side, Saito's theories of primitive form [8] and the higher residue pairing [9] generate the structure of a Frobenius manifold on the singularity's universal deformation space [10], which yield the the genus-0 theory of Landau-Ginzburg B-model. Also, some Hodge theoretical aspect in LG B-model, called the $tt^*$-geometry, is discovered by Cecotti-Vafa [11], whose integrability structure is studied by Dubrovin [12]. Later, Hertling [13] carefully researched the $tt^*$ geometric structure and organized a variety of known structures into the so-called TERP structure. Additionally, Fan [14] introduced an analytic approach in the spirit of $N = 2$ geometry by studying the spectral theory of a twisted Laplacian operator. Besides, Li-Wen [15] used the $L^2$ Hodge theory to give a Frobenius manifold structure for the case $f$ with compact critical locus. Furthermore, motivated by the Virasoro equations and localization calculation in the A-model at a higher genus, Givental [16] gave a remarkable formula for the partition function in the semi-simple case.

The Calabi-Yau/Landau-Ginzburg correspondence connects nonlinear sigma models on Calabi-Yau manifolds with Landau-Ginzburg models. It turns out that CY/LG correspondence and mirror symmetry have served as guidelines in the study of many branched of mathematics (see the following diagram):

$$\begin{tikzcd}
\boxed{\text{Calabi-Yau A-model}} \arrow[r, leftrightarrow, "\text{CY/LG}"] \arrow[d, leftrightarrow, "\text{Mirror Symmetry}"] & \boxed{\text{Landau-Ginzburg A-model}} \arrow[d, leftrightarrow, "\text{Mirror Symmetry}"'] \\
\boxed{\text{Calabi-Yau B-model}} \arrow[r, leftrightarrow, "\text{CY/LG}"'] & \boxed{\text{Landau-Ginzburg B-model}}
\end{tikzcd}$$

We attempt to understand the LG B-model and LG/CY correspondence from an index theoretic perspective. In [3], Bershadshy-Cecotti-Ooguri-Vafa (BCOV) showed that the genus-one term $F_1$ for Sigma model on CY manifolds is computed by some analytic torsion (BCOV torsion) and obtained a holomorphic anomaly formula. In the spirit of the LG/CY correspondence, one might conjecture that the genus-one term for the LG model could be expressed as a torsion-type invariant (We call it BCOV type torsion).

The CY/LG correspondence for the $A$ model is partially done by Chiodo-Iritani-Ruan [17]. The question of whether the LG/CY correspondence holds true for the B-model is an intriguing and challenging one. To address this question, we should understand Witten deformation on noncompact manifolds. Here the Witten deformation $d_f$ is introduced in the extremely influential paper [18], by considering the new differential $d_f = d+df$, where $d$ is the conventional exterior derivative on forms, and $f$ is a Morse function. Setting

$$d_{Tf} := d + Tdf,$$

Witten observed that the eigenfunctions of the small eigenvalues for the corresponding deformed Hodge-Laplacian, the so-called Witten Laplacian, concentrate at the critical points of $f$. As a result, Witten deformation builds a direct bridge between the Betti numbers and the Morse indices of the critical points of $f$.

Witten deformation on closed manifolds has resulted in a wealth of beautiful applications, ranging from Demailly's holomorphic Morse inequalities[19], to the proof of Ray-Singer conjecture and its generalization by Bismut-Zhang [20], to the instigation of the development of Floer homology theory.

The analysis of elliptic and parabolic PDEs on noncompact manifolds is critical in Part I of this thesis, as it leads to an understanding of the asymptotic growth of eigenvalues and decay of eigenfunctions near infinity(Chapter 3), as well as the expansion and the estimate of the heat kernel for Schördinger type operators on noncompact manifolds (Chapter 4). Moreover, these analyses, for example, pave the way for connecting the $L^2$ cohomology of the Witten deformation on noncompact manifolds (Quantum vacuum space of LG models) to other cohomologies, allowing for the definition of important geometric/topological invariants such as the Ray-Singer analytic torsion for Witten deformation on noncompact manifolds (Chapter 5).

Following that, we move to the complex setting and examine the LG/CY correspondence for the B-model. We establish LG/CY correspondence for genus 0 theory in B-model ($tt^*$ structures) structures in Chapter 6. In fact, Carlson-Griffiths [21] compared the cup product and symplectic pairing on the CY and LG sides. Also, Cecotti [22] investigated the $tt^*$ geometry structure on both sides from a physical standpoint. Besides, Fan-Lan-Yang [23] partially proves that the two $tt^*$ structures are isomorphic via the CY/LG correspondence except for the real structures. Here we use different methods than [23, 24] to show the full CY/LG correspondence: we introduce two $U(1)$ actions (called $U(1)$ charges in the physical literatures), which act as certain bi-grading for LG B-models. We also establish CY/LG correspondence for Weil-Peterson type metrics using these two $U(1)$ actions. Furthermore, the Agmon estimate derived in [25] plays an essential role in our method.

To investigate the LG/CY correspondence for the genus 1 term in B-theory, we must generalize our discussion in Chapter 4 and Chapter 5 to families of LG models and the complex/holomorphic setting, which remains unexplored.

## 1.2 Definition and Notations

### 1.2.1 Tameness Conditions

Since we are working on noncompact manifolds, several tameness conditions are needed.

**Definition 1.2.1** (Bounded Geometry). *Let $(M, g)$ be an $n$-dimensional noncompact connected complete Riemannian manifold with metric $g$. $(M, g)$ is said to have bounded* *geometry, if the following conditions hold:*

1. *the injectivity radius $r_0$ of $M$ is positive.*

2. *$|\nabla^m R| \leq C_m$, where $\nabla^m R$ is the $m$-th covariant derivative of the curvature tensor and $C_m$ is a constant only depending on $m$.*

On such a manifold, the Sobolev constant is uniformly bounded, see e.g. [26]. Now let $f : M \to \mathbb{R}$ be a smooth function. In [14], several notions of tameness for the triple $(M, g, f)$ are introduced.

**Definition 1.2.2** (Strongly Tameness)**.** *The triple $(M, g, f)$ is said to be strongly tame, if $(M, g)$ has bounded geometry and*

$$\limsup_{p\to\infty} \frac{|\nabla^2 f|(p)}{|\nabla f|^2(p)} = 0,$$

*and*

$$\lim_{p\to\infty} |\nabla f| \to \infty,$$

*where $\nabla f, \nabla^2 f$ are the gradient and Hessian of $f$ respectively.*

**Remark 1.2.3.** *Fix $p_0 \in M$, and let $d$ be the distance function induced by $g$. Here $p \to \infty$ simply means that $d(p, p_0) \to \infty$.*

**Definition 1.2.4** (Well Tameness)**.** *The triple $(M, g, f)$ is said to be well tame, if $(M, g)$ has bounded geometry and*

$$\limsup_{p\to\infty} \frac{|\nabla^2 f|(p)}{|\nabla f|^2(p)} < \infty,$$

*and*

$$\liminf_{p\to\infty} |\nabla f| > 0.$$

As usual, the metric $g$ induces a canonical metric (still denote it by $g$) on $\Lambda^*(M)$, which then defines an inner product $(\cdot,\cdot)_{L^2}$ on $\Omega_c^*(M)$:

$$(\phi,\psi)_{L^2} = \int_M (\phi,\psi)_g dvol, \phi,\psi \in \Omega_c^*(M).$$

Let $L^2\Lambda^*(M)$ be the completion of $\Omega_c^*(M)$ with respect to $\|\cdot\|_{L^2}$, and for simplicity, we denote $L^2(M) := L^2\Lambda^0(M)$.

For any $T \geq 0$, let $d_{Tf} := d + Tdf\wedge : \Omega^*(M) \to \Omega^{*+1}(M)$ be the so-called Witten deformation of de Rham operator $d$. It is an unbounded operator on $L^2\Lambda^*(M)$ with domain $\Omega_c^*(M)$. Also, $d_{Tf}$ has a formal adjoint operator $\delta_{Tf}$, with $\mathrm{Dom}(\delta_{Tf}) = \Omega_c^*(M)$, such that

$$(d_{Tf}\phi,\psi)_{L^2} = (\phi,\delta_{Tf}\psi)_{L^2}, \phi,\psi \in \Omega_c^*(M).$$

Set $\Delta_{H,Tf} = (d_{Tf} + \delta_{Tf})^2$, and we denote the Friedrichs extension of $\Delta_{H,Tf}$ by $\Box_{Tf}$. As we will see (Theorem 3.1.1), if $(M,g,f)$ is well tame, then $\Delta_{H,Tf}$ is essentially self-adjoint (and hence $\Box_{Tf}$ is the unique self-adjoint extension). In Appendix A (also see Theorem 3.1.3), we will prove the Hodge-Kodaira decomposition when $(M,g,f)$ is well tame and $T$ large enough,

$$L^2\Lambda^*(M) = \ker\Box_{Tf} \oplus \mathrm{Im}\bar{d}_{Tf} \oplus \mathrm{Im}\bar{\delta}_{Tf}, \tag{1.1}$$

where $\bar{d}_{Tf}$ and $\bar{\delta}_{Tf}$ are the minimal extensions of $d_{Tf}$ and $\delta_{Tf}$ respectively.

Setting $\Omega_{(2)}^*(M) := L^2\Lambda^*(M) \cap \Omega^*(M)$, we have a chain complex (of unbounded operators)

$$\cdots \xrightarrow{d_{Tf}} \Omega_{(2)}^*(M) \xrightarrow{d_{Tf}} \Omega_{(2)}^{*+1}(M) \xrightarrow{d_{Tf}} \cdots .$$

Let $H_{(2)}^*(M, d_{Tf})$ denote the cohomology of this complex. In Appendix A, we will show that $H^*_{(2)}(M, d_{Tf}) \cong \ker \Box_{Tf}$, provided $(M, g, f)$ is well tame and $T$ is large enough.

Finally, we note the following well-known fact(Cf. [18, 27]).

**Proposition 1.2.5.** *The Hodge Laplacian $\Delta_{H,Tf}$ has the following local expression:*

$$\Delta_{H,Tf} = \Delta + T\nabla^2_{e_i,e_j} f[e^i \wedge, \iota_{e_j}] + T^2 |\nabla f|^2. \tag{1.2}$$

*Here $\{e_i\}$ is a local frame on $TM$ and $\{e^i\}$ is the dual frame on $T^*M$, $\Delta$ is the usual Hodge Laplacian.*

We now introduce several stronger tameness conditions we need in treating the local index theorem.

**Definition 1.2.6** ($\kappa$-regular Tameness)**.** *Let $(M, g)$ be a complete Riemannian manifold with bounded geometry, and $\kappa \in [0, 1)$. We say $(M, g, f)$ is $\kappa$-regular tame if*

1. $\limsup_{p\to\infty} \frac{|\nabla^m f|}{|\nabla f|^{(m-1)\kappa+1}} < \infty$, *for any* $m \geq 1$;
2. $\lim_{p\to\infty} |\nabla f| = \infty$.

In typical examples from Landau-Ginzburg models, $M = \mathbb{C}^n$ with the Euclidean metric and a nondegenerate quasi-homogeneous polynomial $f$. Then $(\mathbb{C}^n, f)$ is $\kappa$-regular tame for some $\kappa < 1$, see the discussion in the last chapter. For our purpose, we reformulate one of the consequences of $\kappa$-regular tameness. Indeed, an inductive argument yields that, if $(M, g, f)$ is $\kappa$-regular tame, then for $V = |\nabla f|^2$, we have, for all $k \in \mathbb{N}$,

$$\limsup_{p\to\infty} \frac{|\nabla^k V|}{|V|^{(k\kappa+2)/2}} < \infty.$$

**Remark 1.2.7.** *In [28], in order to prove Weyl's law for Schrödinger operator on $\mathbb{R}^n$, Rozenbljum imposed similar $\kappa$-tameness conditions (see (0.6) in [28]). In the appendix,* *we will show that with $\kappa$-tameness condition, one can prove a weaker version of Weyl's law.*

Our next condition ensures that we have a good local index theory.

**Definition 1.2.8** ($\alpha$-polynomial Tameness)**.** *Fix $\alpha \geq n/2$. A triple $(M, g, f)$ is called $\alpha$-polynomial tame, if $(M, g, f)$ is $\kappa$-regular tame for some $\kappa \in (0, 1)$, and in addition, there is some constant $C$, such that for all $\lambda \geq 0$,*

$$\int_{\{p \in M : |\nabla f|^2(p) \leq \lambda\}} (\lambda - |\nabla f|^2)^{n/2} dvol_M \leq C\lambda^{\alpha}.$$

Again we will see that typical examples coming from Landau-Ginzburg models are polynomial tame.

**Remark 1.2.9.** *This condition should be interpreted in terms of the (semiclassical) Weyl's law for Schrödinger operators in Euclidean space ( Cf. [28, 29] ), which would guarantee the polynomial growth of the eigenvalues. However, though expected, we could not find in the literature such Weyl's law on manifolds. To focus our discussion on the asymptotic expansion of heat kernel and local index theorem, we only prove a weaker version of Weyl's law here under our assumption.*

### 1.2.2 $tt^*$ Structures

To analyze the holomorphic anomaly formula and the CY/LG correspondence for genus-one terms, we investigate the $tt^*$ geometry structure, which captures the 2D vacuum geometry in string theory. People believe that the $tt^*$ structure of the B-model carries all the information of genus-zero terms. We will see later that both the CY B-model and the LG B-model have $tt^*$ structures.

Here we briefly illustrate the concept of $tt*$ geometry; interested readers can refer to [30] for more information: Let $\mathcal{H}$ be the Hilbert space of a QFT. Assume $V \subset \mathcal{H}$ is a distinct subspace with a fixed finite dimension. For instance, let $\mathcal{V}$ be the space of ground states $|0\rangle$. Assume we have a family of physical theories parametrized by the moduli space $\mathcal{M}$ and the full Hilbert space of physical theories stays unchanged, i.e., the Hilbert space $\mathcal{H}$ is a trivial bundle over $\mathcal{M}$. The distinguished subspace $V(m)$, on the other hand, now depends on $m \in \mathcal{M}$. If $\mathcal{V} = \cup_{m\in\mathcal{M}} V(m)$, then $\mathcal{V} \to \mathcal{M}$ is a vector bundle.

In the case of $(2,2)$ theories, it will turn out that $\mathcal{M}$ carries a Kähler structure, and $\mathcal{V} \to \mathcal{M}$ is a holomorphic vector bundle, then we could assume $m = (t, \bar{t})$.

Let $\left|\alpha\left(m_i\right)_j\right\rangle$ be an orthonormal basis of $V(m)$, i.e.,

$$\langle \alpha(m)_k \mid \alpha(m)_j \rangle = \delta_{jk}$$

The triviality of the Hilbert space bundle over $\mathcal{M}$ naturally defines a connection $D$ on the $V$-bundle,

$$(A_i)_j^k = \left\langle \alpha(m)_k \left| \frac{\partial}{\partial m_i} \right| \alpha(m)_j \right\rangle .$$

Moreover, fix $m \in \mathcal{M}$, follows from state-operator correspondence in CFT, one has Kodaira-Spencer type map $C : T_m\mathcal{M} \mapsto End(V_m)$. It follows from some physical arguments (not rigorous in mathematics since path integral is involved) that

$$\begin{gathered}
[D_i, D_j] = 0, \\
\left[\bar{D}_i, \bar{D}_j\right] = 0, \\
[D_i, C_j] = [D_j, C_i], \quad \left[\bar{D}_i, \bar{C}_j\right] = \left[\bar{D}_j, \bar{C}_i\right], \\
\left[D_i, \bar{D}_j\right] = -\left[C_i, \bar{C}_j\right],
\end{gathered} \tag{1.3}$$

where $D_i = D_{\partial_{t_i}}, \bar{D}_i = D_{\partial_{\bar{t}_i}}, C_i = C(\partial_{t_i}), \bar{C}_i = C(\partial_{\bar{t}_i})$. As a result, $\nabla^z = D + \bar{D} + \frac{1}{z}C + z\bar{C}$ is a flat connection for any $z \in \mathbb{C}^*$.

Lastly, $\mathcal{V} \to \mathcal{M}$ carries a natural real structure $\kappa$ induced from $\mathcal{H} \to \mathcal{M}$, which satisfies several nice properties that we will state in a moment.

In summary, one has

**Definition 1.2.10** ($tt^*$ structures). *Let $M$ be a complex manifolds, a $tt^*$ geometry structure $(K \to M, \kappa, g, D, \bar{D}, C, \bar{C})$ consists of the following data:*

- *$K \to M$ is a holomorphic vector bundle,*
- *a complex anti-linear involution $\kappa : K \to K$, i.e. $\kappa^2 = \mathrm{Id}, \kappa(\lambda\alpha) = \bar{\lambda}\kappa(\alpha), \forall \lambda \in \mathbb{C}$,*
- *a Hermitian metric $g(u, v)$,*
- *a one parametric family of flat connections $\nabla^z = D + \bar{D} + \frac{1}{z}C + z\bar{C}$, where $D + \bar{D}$ is the Chern connection of $g$, $D$ is the $(1, 0)$ component of Chern connection, $\bar{D}$ is the $(0, 1)$ component of Chern connection, $C, \bar{C}$ are the $C^\infty(M)$-linear map*

$$C : C^\infty(K) \to C^\infty(K) \otimes \mathcal{A}_M^{(1,0)}, \quad \bar{C} : C^\infty(K) \to C^\infty(K) \otimes \mathcal{A}_M^{(0,1)}$$

  *satisfying*

  1. *$g$ is real with respect to $\kappa : g(\kappa(u), \kappa(v)) = \overline{g(u, v)}$,*
  2. *$(D + \bar{D})(\kappa) = 0, \bar{C} = \kappa \circ C \circ \kappa$,*
  3. *$\bar{C}$ is the adjoint of $C$ with respect to $g$, i.e. $g(C_X u, V) = g\left(u, \bar{C}_X v\right), X \in \mathcal{T}_M$.*

*Then we say such structure $(K \to M, \kappa, g, D, C, \bar{C})$ is a $tt^*$ geometry structure.*

**Definition 1.2.11** (Morphism of $tt^*$ structures). *Let $G_i = \left(K_i \longrightarrow M, \kappa_i, \eta_i, D_i, C_i, \bar{C}_i\right), i = 1, 2$, be two $tt^*$ geometric structures. An morphism $\phi : K_1 \mapsto K_2$ of two holomorphic bun-*

*dles is called a morphism of two $tt^*$ geometric structure $G_1$ and $G_2$ if the following hold:* $\forall p \in M, X \in T_pM, u, v \in (H_1)_p$,

1. $\eta_1(u,v) = \eta_2(\phi(u), \phi(v))$.

2. $\kappa_2 \circ \phi = \phi \circ \kappa_1$.

3. $\phi\left((D_1)_X u\right) = (D_2)_X (\phi(u))$ *and* $\phi \circ \left(\left(\bar{D}_1\right)_{\bar{X}} (\kappa_1(u))\right) = \left(\bar{D}_2\right)_{\bar{X}} (\kappa_2(\phi(u))$

4. $\phi\left((C_1)_X u\right) = (C_2)_X (\phi(u))$.

*Two $tt^*$ structures are isomorphic if there exists morphism $\phi : G_1 \to G_2$, $\psi : G_2 \to G_1$, s.t.* $\phi \circ \psi = \mathrm{Id}_{G_2}$, $\psi \circ \phi = \mathrm{Id}_{G_1}$.

### 1.2.3 Mixed Hodge Structures and $tt^*$ Structures on LG B-models

#### Mixed Hodge Structures on LG Models

Following Steenbrink [31], we explore the mixed Hodge structures of LG B-model for a quasi-homogeneous polynomials: we consider a pair $(X = \mathbb{C}^n, f_0)$, where $f : \mathbb{C}^n \to \mathbb{C}$ is a quasi-homogenous polynomial, i.e., there exist $q_1, \ldots, q_n \in \mathbb{Q}$ such that for any $\lambda \in \mathbb{C}^*$,

$$f(\lambda^{q_1} z_1, \ldots, \lambda^{q_n} z_n) = \lambda f(z_1, ..., z_n).$$

Each $q_i$ is called the weight of $z_i$, denote by $\mathrm{wt}(z_i) = q_i$.

While, for convenience, we assume that $f$ is non-degenerate, i.e. we require that

1. $f$ contains no monomial of the form $z_i z_j$ for $i \neq j$,

2. $f$ has only an isolated singularity at the origin.

The polynomial $f$ plays the role of superpotential for LG model. For any $t \in \mathbb{C}^*$, $V_t := \{z \in \mathbb{C}^n : f(z) = t\}$. Then we consider a projective compactification of $V_1$ in a weighted projective space. Let $q_i = a_i/b_i$ with $(a_i, b_i) = 1$, and $d = \mathrm{lcm}\,(b_1, \ldots, b_n)$, i.e. $d$ is the least common multiple of $b_1, ..., b_n$. We put $Q_i = q_i d$. Let $z = (z_1, ..., z_n)$, and $Z = (z, z_{n+1})$, then the polynomial

$$F_t(Z) \equiv f\,(z) - tz_{n+1}^d.$$

is also quasi-homogeneous. Let $M = \mathbf{P}\,(Q_1, Q_2, \ldots, Q_n, 1)$ be the weight projective space with weights $(Q_1, Q_2, ..., Q_n, 1)$, then $\bar{V}_t := \{[Z] \in \mathbf{P}\,(Q_1, Q_2, \ldots, Q_n, 1) : F_t(Z) = 0\}$. If we identify $C^n$ with the open set in $M$ where $x_{n+1} \neq 0$, we see that $V_t = \bar{V}_t \cap C^n$. Generically, $\bar{V}_t$ is not smooth: it is an orbifold. Let $M_\infty$ be the hypersurface in $M$ given by $X_{n+1} = 0$. Then $M_\infty \cong \mathbf{P}\,(Q_1, Q_2, \ldots, Q_n)$. Consider the part of $\bar{V}_t$ at infinity,

$$V_\infty \equiv \bar{V} \backslash V_t = \bar{V} \cap M_\infty,$$

i.e. the hypersurface in $M_\infty$ given by the equation

$$f(z) = 0.$$

Again, $V_\infty$ is an orbifold (in general). It follows from the Mayer-Vietoris exact sequence of the couple $\left(\bar{V}_t, V_t\right)$ and the Thom isomorphism $H^*\left(\bar{V}_t, V_t\right) \simeq H^{*-2}\,(V_\infty)$ that

$$\cdots \to H^i(\bar{V}_t) \to H^i\,(V_t) \to H^{i-1}\,(V_\infty) \to H^{i+1}(\bar{V}_t) \to \cdots \tag{1.4}$$

Hence

$$H^{n-1}\,(V_t) \simeq H^{n-1}(\bar{V}_t)_0 \oplus H^{n-2}\,(V_\infty)_0\,,$$

where

$$H^{n-1}(\bar{V})_0 = \text{coker}\left(H^{n-3}\left(V_\infty\right) \to H^{n-1}(\bar{V}_t)\right),$$
$$H^{n-2}\left(V_\infty\right)_0 = \ker\left(H^{n-2}\left(V_\infty\right) \to H^{n}(\bar{V}_t)\right).$$

From the Lefschetz theorem it is clear that $H^{n-2}\left(V_\infty\right)_0$ is just the primitive cohomology of the Kähler orbifold $V_\infty$. The weight graduation of $H^{n-1}\left(V_t\right)$ is

$$\begin{aligned} \mathrm{Gr}_{n-1}H^{n-1}\left(V_t\right) &\simeq H^{n-1}(\bar{V})_0 \\ \mathrm{Gr}_{n}H^{n-1}\left(V_t\right) &\simeq H^{n-2}\left(V_\infty\right)_0[-1]. \end{aligned} \tag{1.5}$$

Hence $H^{n-1}(\bar{V})_0$ and $H^{n-2}\left(V_\infty\right)$ carry pure Hodge structures which define the Hodge filtration of the group $H^{n-1}\left(V_t\right)$. In (1.5), $[-1]$ means "Tate twist", i.e., a degree shift by $-1$ in the Hodge filtration:

$$\mathscr{F}^p\mathrm{Gr}_nH^{n-1}\left(V_t\right) \simeq \mathscr{F}^{p-1}H^{n-2}\left(V_\infty\right)_0,$$

which defines a morphism of Hodge structures of type $(-1,-1)$.

Let $\Omega^k(\mathbb{C}^n)$ be the space of holomorphic $k$-forms, $H^k_{(2)}(\mathbb{C}^n,\bar{\partial}_f)$ be the $k$-th $L^2$-cohomology w.r.t. Witten deformation $\bar{\partial}_f := \bar{\partial} + \partial f\wedge$, $\mathrm{Jac}(f) := \mathbb{C}[z_1,...,z_n]/<\partial_1 f,...,\partial_n f>$, then one can show that (c.f. [25, 23, 14]), as vector spaces,

$H^{n-1}(V_t) \simeq H^n_{(2)}(\mathbb{C}^n,\bar{\partial}_f) \simeq \Omega^n(\mathbb{C}^n)/df\wedge\Omega^{n-1}(\mathbb{C}^n) \simeq \mathrm{Jac}(f)$.

Now we would like to describe mixed Hodge structure on $\Omega^n(\mathbb{C}^n)/df\wedge\Omega^{n-1}(\mathbb{C}^n) \simeq \mathrm{Jac}(f)$ directly.

Fix a homogeneous basis $\{\phi_a\}_{a=0,\dots,\mu-1}$ of $\mathrm{Jac}(f)$ with $\phi_0 = 1$, $\deg(\phi_a) \le n-2$, and let $A_a := \phi_a dz_1...dz_n \in \Omega^n(\mathbb{C}^n)/df\wedge\Omega^{n-1}(\mathbb{C}^n)$.

**Definition 1.2.12.** *For $A = z_1^{\beta_1} \cdots z_n^{\beta_n} dz_1 \wedge \cdots \wedge dz_n$, we define*

$$l(A) := \sum_{i=1}^{n} (\beta_i + 1) Q_i.$$

**Now assume that $f$ is homogeneous of degree $n$.** In this case, $Q_i \equiv 1$.

Let $\mathrm{Jac}(f)' := \mathrm{Span}\{\phi_a : n | l(A_a)\}$, then by [31], as a vector space

$$\mathrm{Gr}_n H^{n-1}(V_t) \simeq \mathrm{Jac}(f)'.$$

Moreover, one also has

$$\mathscr{F}^p \mathrm{Gr}_n H^{n-1}(V_t) \simeq \mathrm{Span}\{\phi_a \in \mathrm{Jac}(f)' : l(A_a) \leq n(n-1-p)\}.$$

In Section 6.3, we will define a map $R : \mathrm{Jac}(f)' \to H^{n-2}(V_\infty)_0[-1]$ that preserve the filtration. In particular, $R(1) \in \Omega^{n-2,0}(V_\infty)$.

**Monodromy, Leray Coboundary Map and Lefschetz Thimble.**

Let $c_t(s) := te^{2\pi i s}$, $0 \leq s \leq 1$, then $c_t$ induces a monodromy operator $M : H_{n-1}(V_t) \to H_{n-1}(V_t)$.

Dualize (1.4), one shows

$$\cdots \to H_n(\bar{V}_t) \to H_{n-2}(V_\infty) \xrightarrow{\tau} H_{n-1}(V_t) \to H_{n-1}(\bar{V}_t) \to \cdots,$$

where $\tau$ is the Leray coboundary map.

Then following Steenbrink [31], one has

$$\mathrm{Gr}_n H^{n-1}(V_t)^* \simeq \ker(M - \mathrm{Id}) \simeq \tau(H),$$

where $V^*$ denotes the dual space of a vector space $V$.

Let $s > 0$, and consider the sets

$$F^{\geq s} = \{z \in \mathbb{C}^n \mid \mathrm{Re}(F(z,u)) \geq s\}\,, \quad F^{\leq -s} = \{z \in \mathbb{C}^n \mid \mathrm{Re}(F(z,u)) \leq -s\}\,.$$

By Morse theory, we know that if there is no critical values of $\mathrm{Re}(F(z,u))$ between $[a,b]$, then the set $f^{\leq a}$ is the deformation kernel of $F^{\leq b}$ by the flow generated by the vector field $\nabla\,\mathrm{Re}(F(z,u))/|\nabla\,\mathrm{Re}(F(z,u))|$. So if $s$ is large enough, there is no critical points in $F^{\leq -s}$ and each $F^{\leq -s}$ has the same homotopy type. We denote their equivalence class by $F^{-\infty}$. Similarly, we have $F^{+\infty}$. We have the relative homology group $H_*\left(\mathbb{C}^n, F^{-\infty}; \mathbb{Z}\right)$ and $H_*\left(\mathbb{C}^n, F^{+\infty}; \mathbb{Z}\right)$.

Following [23], we fix a basis $\{\sigma_k\}_{k=0}^{\mu-1}$ of $H_{n-1}(V_{-1})$, such that $\sigma_k \in \ker(M - \mathrm{Id})$ for $0 \leq k \leq \mu' - 1$. Hence, there exists $\delta_k \in H_{n-2}(V_\infty)_0 := (H^{n-2}(V_\infty)_0)^*$, s.t. $\sigma_k = \tau(\delta_k)$ for $0 \leq k \leq \mu' - 1$. For $t > 0$, let $\Phi_t : V_{-t} \to V_{-1}$ be the map $(z_1, .., z_n) \to (t^{-\frac{1}{n}} z_1, ..., t^{-\frac{1}{n}} z_n)$, and set

$$\gamma_k := \cup_{t>0} (\Phi_t)^* \sigma_k,$$

then one can see easily that $\{\gamma_k\}_{k=0}^{\mu-1}$ is a basis of $H_n(\mathbb{C}^n, f^{-\infty})$ (called Lefchetz thimble).

Similarly, one can construct a basis $\{\tilde{\gamma}_k\}_{k=0}^{\mu-1}$ of $H_n(\mathbb{C}^n, f^{+\infty})$.

Now we set up some notations for intersection matrices: let $(I)_{ij}(0 \leq i, j \leq \mu - 1)$ be the intersection matrix (see Definition 6.4.1 for the definition), $\mathcal{I} = I^{-1}$; $I'_{ij}(0 \leq i, j \leq \mu' - 1)$ be a submatrix of $I$, and $\mathcal{I}' = (I')^{-1}$; $(I^{CY})_{ij} = \delta_i \cap \delta_j$ be the intersection matrix, and $\mathcal{I}^{CY} = (I^{CY})^{-1}$.

**$tt^*$ Structures on LG Models**

For simplicity, consider a marginal deformation of $f$:

$$F(u,x) = f(x) + \sum_{i=1}^{\nu} u^i \psi_i(x),$$

i.e. $l(\psi_i dz_1 ... dz_n) = l(f dz_1 ... dz_n)$, $i = 1, \ldots, \nu$. We denote $M$ by the space of parameters $u^i$, which should be a small neighborhood of the origin in $\mathbb{C}^\nu$. As a result, we have a family of supersymmetric algebra operators $\bar{\partial}_F, \partial_F, \bar{\partial}_F^*, \partial_F^*, \Delta_F$ parameterized by $u \in M$.

First, we have the following trivial complex Hilbert bundle $L^2\Lambda^*(X) \times M \to M$. For simplicity, denote by $L^2\mathcal{A}$ its $L^2$-integrable section space. There are two natural parings,

$$\begin{aligned} h:\ L^2\mathcal{A} \times L^2\mathcal{A} &\longrightarrow C^\infty(M) \qquad h(\alpha,\beta) = \int_X \alpha \wedge *\bar{\beta}, \\ \eta:\ L^2\mathcal{A} \times L^2\mathcal{A} &\longrightarrow C^\infty(M) \qquad \eta(\alpha,\beta) = \int_X \alpha \wedge *\beta. \end{aligned}$$

Moreover, there is a canonical real structure on the Hilbert bundle, which is given by the complex conjugate $\tau_R$. Then $h(\alpha,\beta) = \eta(\alpha, \tau_R \beta)$

Let $H^n := H_F^n$ be the Hodge bundle over $M$, and its fiber at $u \in M$ is the space of all harmonic $n$-forms of $\Delta_{F(u)}$. We denote the space of its section by $\mathcal{H}$.

Let $\Pi_u : L^2\mathcal{A} \to \mathcal{H}$ be the harmonic projection, $G$ be the inverse operator of $\Delta_F$ on $\mathrm{im}(\bar{\partial}_F) \oplus \mathrm{im}(\bar{\partial}_F^*)$, then $G$ commutes with the operators $\bar{\partial}_F, \partial_F, \bar{\partial}_F^*, \partial_F^*, \Delta_F$, and the operator form of Hodge decomposition reads

$$\mathrm{Id} = \Pi_u + \Delta_F G = \Pi + G\Delta_F.$$

On the Hodge bundle, we have

(1) The connection $D$, $\bar{D}$

Notice that the Hodge bundle is embedded into the Hilbert bundle, so we can define $D$, $\bar{D}$ in a natural way:

$$D_i = \Pi_u \circ \partial_i, \quad \bar{D}_{\bar{i}} = \Pi_u \circ \bar{\partial}_{\bar{i}} \quad i = 1, ..., s.$$

(2) The operators $C_i, \bar{C}_{\bar{i}}$

We define $C_i = \Pi_u \circ \partial_i F = \Pi_u \circ \psi_i$, $\bar{C}_{\bar{i}} = \Pi_u \circ \overline{\partial_i F} = \Pi_u \circ \overline{\psi_i}$. We can also compute that

$$C_i = (\partial_i F) - \bar{\partial}_F \bar{\partial}_F^* G(\partial_i F), \quad \bar{C}_{\bar{i}} = (\overline{\partial_i F}) - \partial_F \partial_F^* G(\overline{\partial_i F}).$$

By definition, $\bar{C}_{\bar{i}}$ is the adjoint operator of $C_i$ with respect to the $tt^*$ metric $h$, i.e.

$$h(C_i \alpha, \beta) = h(\alpha, \bar{C}_{\bar{i}} \beta).$$

**Proposition 1.2.13** ($tt^*$ equation)**.** *The operators $D_i, \bar{D}_{\bar{j}}$, $C_i, \bar{C}_{\bar{j}}$ satisfy the following equations*

1. $[C_i, C_j] = 0, \quad [\bar{C}_{\bar{i}}, \bar{C}_{\bar{j}}] = 0, \quad [D_i, \bar{C}_{\bar{j}}] = [\bar{D}_{\bar{i}}, C_j] = 0;$
2. $[D_i, C_j] = [D_j, C_i], \quad [\bar{D}_{\bar{i}}, \bar{C}_{\bar{j}}] = [\bar{D}_{\bar{j}}, \bar{C}_{\bar{i}}];$
3. $[D_i, D_j] = 0, \quad [\bar{D}_{\bar{i}}, \bar{D}_{\bar{j}}] = 0, \quad [D_i, \bar{D}_{\bar{j}}] = -[C_i, \bar{C}_{\bar{j}}].$

As a result, $(\mathcal{H} \to M, \tau_R, h, D, C, \bar{C})$ carries a $tt^*$ structure.

**Now assume that $f$ is homogeneous of degree $n$.**

**Definition 1.2.14** (small $tt^*$ structure on LG models)**.** *Let $\mathcal{H}' \subset \mathcal{H}$ be the subbundle generated by $w_{k_a}$, where $l(A_{k_a})/n \in \mathbb{Z}$. By Proposition 6.2.1, restriction of $\tau_R, h, D, C, \bar{C}$ to $\mathcal{H}'$ defines a $tt^*$ structure, called small $tt^*$ geometry structure on LG models.*

## 1.3 Main Results

### 1.3.1 On Agmon Estimate and Thom-Smale-Witten Complex

In this subsection, we assume that $(M, g)$ has bounded geometry, $f$ is a Morse function with finite many critical points. Clearly this will be the case if $(M, g, f)$ is well tame and $f$ is Morse.

As we mentioned the main technical result here is the Agmon estimate for the eigenforms of the Witten Laplacian.

**Theorem 1.3.1.** *Let $(M, g, f)$ be well tame, and $\omega \in \mathrm{Dom}(\Box_{Tf})$ be an eigenform of $\Box_{Tf}$ whose eigenvalue is uniformly bounded in $T$. Then*

$$|\omega(p)| \leq CT^{(n+2)/2} \exp(-a\rho_T(p))\|\omega\|_{L^2},$$

*for any $a \in (0, 1)$ (provided $T$ is sufficiently large and $C$ is a constant depending on the dimension $n$, the function $f$, the curvature bound, the injectivity radius lower bound $r_0$, and $a$; for the precise choice of $T, C$ see the end of Section 3). Here the definition of the Agmon distance $\rho_T(p)$ will be given in Section 3.2.*

The proof of the Agmon estimate, given in Section 3.5, is to carry out the idea of [32] in this more general setting.

Set $b_i(T) = \dim H^i_{(2)}(M, d_{Tf})$. If $x$ is a critical point of $f$, denote $n_f(x)$ the Morse index of $f$ at $x$. Let $m_i$ be the number of critical points of $f$ with Morse index $i$. Then the strong Morse inequalities hold.

**Theorem 1.3.2.** *If $(M, g, f)$ is well tame, then we have the following strong Morse inequality*

$$(-1)^k \sum_{i=0}^{k} (-1)^i b_i(T) \leq (-1)^k \sum_{i=0}^{k} (-1)^i m_i, \quad \forall k \leq n,$$

*provided $T$ is large enough. And the equality holds for $k = n$.*

In general, $b_i(T)$ may be very sensitive to $T$. However, we have the following result regarding the independence of $b_i(T)$ in $T$. Assume that the Morse function $f$ satisfies the Smale transversality condition. Let $(C^*(W^u), \tilde{\partial}')$ be the Thom-Smale complex given by $f$. It is important to note that in general, since $M$ is noncompact, it could happen that $(\tilde{\partial}')^2 \neq 0$. Also let $c > 0$ be big enough, $U_c = \{p \in M : f(p) < -c\}$ and $(\Omega^*(M, U_c), d)$ be the relative de Rham complex.

**Theorem 1.3.3.** *If $(M, g, f)$ is well tame, then $(\tilde{\partial}')^2 = 0$, and therefore the cohomology $H^*(C^\bullet(W^u), \tilde{\partial}')$ is well defined. Moreover, there exists $T_0 \geq 0$, such that $H^*_{(2)}(M, d_{Tf})$ is isomorphic to $H^*(C^\bullet(W^u), \tilde{\partial}')$ for all $T > T_0$. In addition, $H^*(C^\bullet(W^u), \tilde{\partial}')$, hence $H^*_{(2)}(M, d_{Tf})$ is isomorphic to the relative de Rham cohomology $H^*_{dR}(M, U_c)$.*

**Remark 1.3.1.** *When $(M, g, f)$ is strongly tame, $T_0 = 0$.*

By Theorem 1.3.3, we can refine our result of Theorem 1.3.2.

**Corollary 1.3.4.** *If $(M, g, f)$ is well tame, then $b_i(T)$ is independent of $T$ when $T$ is big enough. When $(M, g, f)$ is strongly tame, $b_i(T)$ is independent of $T > 0$.*

**Remark 1.3.2.** *Assume that $M$ is oriented and let $*$ be the Hodge star operator. Then $*\Box_{Tf} = \Box_{-Tf}*$. Hence we have the following Poincaré duality:*

$$H^k(M, d_{Tf}) \cong H^{n-k}(M, d_{-Tf}).$$

### 1.3.2 Heat Kernel Expansion and Local Index Theorem

As we will see in the next section, the $\alpha$-polynomial tame condition guarantees that $\exp(-t\Box_{Tf})$ is of trace class. Our first contribution is the pointwise asymptotic expansion of the heat kernel of the Witten Laplacian with a strong remainder estimate.

Let $(M, g, f)$ be $\alpha$-polynomial tame, and $K_{Tf}(t, x, y)$ denote the heat kernel of the Witten Laplacian $\square_{Tf}$. Denote by $h_T(x, y)$ the average of the potential function $T^2|\nabla f|^2$ on the geodesic segment from $x$ to $y$, Cf. (4.8).

**Theorem 1.3.5.** *The heat kernel $K_{Tf}$ has the following complete pointwise asymptotic expansion. For any $x, y \in M$ such that $d(x, y) \leq \frac{1}{2}\tau$,*

$$K_{Tf}(t, x, y) \sim \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp(-d^2(x, y)/4t) \exp(-t h_T(x, y)) \sum_{j=0}^{\infty} t^j \theta_{T,j}(x, y),$$

*as $t \to 0$. Each $\theta_{T,j}$ is a polynomial of $T$:*

$$\theta_{T,j}(x, y) = \sum_{l=0}^{[\frac{j}{3}]+j} T^l \theta_{l,j}(x, y),$$

*and, when restricted to the diagonal of $M \times M$, $\theta_{l,j}(y, y)$ can be written as an algebraic combination of the curvature of the metric $g$, the function $f$, as well as their derivatives, at $y$; in addition, $\theta_{T,0}(y, y) = \mathrm{Id}$. Moreover, we have the following remainder estimate. For any $k$ sufficiently large and any $a \in (0, 1)$,*

$$\left| K_{Tf}(t, x, y) - \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp(-d^2(x, y)/4t) \exp(-t\, h_T(x, y)) \sum_{j=0}^{k} t^j \theta_{T,j}(x, y) \right|$$
$$\leq C t^{\frac{1}{3}(1-\kappa)k - \frac{\kappa+2}{3} - \frac{n}{2} + 1} T^{\frac{-2k+4}{3}} \exp(-a \tilde{d}_T(t, x, y)),$$

*for $t \in (0, 1]$ and $T \in (0, t^{-\frac{1}{2}}]$.*

Here $\tilde{d}_T(t, x, y)$ is the parabolic distance alluded at the beginning of the introduction, see (4.14) for the precise definition. By relating it to the Agmon distance we can obtain an effective bound on $\tilde{d}_T(t, x, y)$, which, when combined with the theorem above, yields the following corollary.

**Corollary 1.3.6.** *For $T = t^{-\frac{1}{2}}$, and any $k$ sufficiently large, any $a \in (0,1)$,*

$$\left| K_{t^{-\frac{1}{2}}f}(t,x,x) - \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp(-|\nabla f|^2(x)) \sum_{j=0}^{k} \sum_{l=0}^{[\frac{j}{3}]+j} t^{j-\frac{l}{2}} \theta_{l,j}(x,x) \right|$$
$$\leq C t^{\frac{1}{3}(2-\kappa)k - \frac{\kappa+1}{3} - \frac{n}{2}} \exp(-a\bar{\beta}|\nabla f|^{1-\kappa}(x)),$$

*for $t \in (0,1]$, where $\bar{\beta} > 0$ is a constant depending only on the bounds in the tameness condition. In particular, we have the following small time asymptotic expansion of the heat trace:*

$$\mathrm{Tr}_s\left(\exp(-t\Box_{t^{-\frac{1}{2}}f})\right) \sim \frac{1}{(4\pi t)^{\frac{n}{2}}} \sum_{j=0}^{\infty} \sum_{l=0}^{[\frac{j}{3}]+j} t^{j-\frac{l}{2}} \int_M \exp(-|\nabla f|^2(x)) \, \mathrm{tr}_s(\theta_{l,j}(x,x)) dx,$$

*as $t \to 0$, with the remainder estimate*

$$\left| \mathrm{Tr}_s\left(\exp(-t\Box_{t^{-\frac{1}{2}}f})\right) - \frac{1}{(4\pi t)^{\frac{n}{2}}} \sum_{j=0}^{k} \sum_{l=0}^{[\frac{j}{3}]+j} t^{j-\frac{l}{2}} \int_M \exp(-|\nabla f|^2(x)) \, \mathrm{tr}_s(\theta_{l,j}(x,x)) dx \right|$$
$$\leq C t^{\frac{1}{3}(2-\kappa)k - \frac{\kappa+1}{3} - \frac{n}{2}}.$$

*Here* $\mathrm{Tr}_s$ *and* $\mathrm{tr}_s$ *denote the global supertrace and pointwise supertrace respectively.*

On the other hand, by Theorem 1.3 in [25] and the $\alpha$-polynomial tame condition, the index of the Witten Laplacian

$$\chi(M, d_{Tf}) = \sum_{i=0}^{n} (-1)^i b_i(T), \;\; b_i(T) = dim(H^i_{(2)}(M, d_{Tf}))$$

is independent of $T > 0$. In fact we have that $\mathrm{Tr}_s(exp(-t\Box_{Tf}))$ is independent of $t$ and

$T$ and

$$\chi(M, d_{Tf}) = \mathrm{ind}(\Box_{Tf}) = \mathrm{Tr}_s(-t\exp(\Box_{t^{-\frac{1}{2}}f})) = \int_M \mathrm{tr}_s(K_{Tf}(t,x,x))dx.$$

Now apply our new rescaling technique, one has

**Theorem 1.3.7** (Local index theorem and index formula for $\Box_{Tf}$)**.** *For any $x \in M$, we have*

$$\lim_{t\to 0} \mathrm{tr}_s(K_{t^{-\frac{1}{2}}f}(t,x,x)) = \frac{(-1)^{[\frac{n+1}{2}]}}{\pi^{\frac{n}{2}}} \exp(-|\nabla f(x)|^2) \int^B \exp(-\frac{\widetilde{R}(x)}{2} - \widetilde{\nabla}^2 f(x)).$$

*In particular, for any $T > 0$,*

$$\mathrm{ind}(\Box_{Tf}) = \frac{(-1)^{[\frac{n+1}{2}]}}{\pi^{\frac{n}{2}}} \int_M \exp(-|\nabla f|^2) \int^B \exp(-\frac{\tilde{R}}{2} - \tilde{\nabla}^2 f). \tag{1.6}$$

Here $\int^B$ denotes the Berezin integral, which will be introduced in a moment, and $\tilde{R}, \tilde{\nabla}^2 f \in \Omega^*(M)\hat{\otimes}\Omega^*(M)$ are defined as

$$\tilde{R} = -\sum_{i<j,k<l} R_{ijkl} e^i e^j \hat{e}^k \hat{e}^l, \qquad \tilde{\nabla}^2 f = \nabla^2_{e_i,e_j} f e^i \hat{e}^j$$

for some orthonormal frame $\{e_i\}$ in $TM$ and its dual frame $\{e^i\}$ in $T^*M$. We have used $\{\hat{e}_i\}$ to denote the same orthonormal frame in the second copy of $T^*M$. For any $\omega \in \Omega^*(M)\hat{\otimes}\Omega^*(TM)$, $I = \{i_1, ..., i_k\} \subset 1, 2, ..., n$, we write $\omega$ as

$$\omega = \sum_I w_I \hat{e}^I,$$

where $\hat{e}^I = \hat{e}^{i_1} \wedge ... \wedge \hat{e}^{i_k}$. Then the Berezin integral is defined as

$$\int^B : \Omega^*(M)\hat{\otimes}\Omega^*(M) \mapsto \Omega^*(M), \qquad \int^B \omega = \omega_{1,2,...,n}.$$

**Remark 1.3.3.**

1. *Here the index density is computed by coupling $tT^2 = 1$. Our arguments still work if we set $tT^2$ to be any positive constant $T_0$. As $T_0 \to \infty$, the integral of index density localizes at critical points of $f$. On the other hand, when $T_0 \to 0^+$, the index of $\Box_{Tf}$ should depend on "the topology away from infinity" and the behavior of $f$ near the infinity. This will be discussed in more detail in a separate paper where we extend our treatment to Dirac/Callias type operators.*

2. *When $M$ is compact, (1.6) is a special case of a formula in Chapter 3 of [27].*

3. *Notice that $\int^B \exp(-\nabla^2 f) = (-1)^{[\frac{n}{2}]} \det(-\nabla^2 f)$. Thus, when $M = \mathbb{R}^n$, (1.6) reduces to*
$$\chi(\mathbb{R}^n, d_f) = \frac{(-1)^n}{\pi^{\frac{n}{2}}} \int_{\mathbb{R}^n} \exp(-|\nabla f|^2) \det(-\nabla^2 f) dvol.$$
*In particular, when $M = \mathbb{C}^n$, $f$ is a holomorphic function such that its real part* Re$f$ *is polynomial tame, we have*
$$\begin{aligned}\chi(\mathbb{C}^n, d_f) &= \frac{1}{\pi^n} \int_{\mathbb{C}^n} \exp(-|\nabla \mathrm{Re} f|^2) det(-\nabla^2 \mathrm{Re} f) dvol \\ &= \frac{(-1)^n}{\pi^n} \int_{\mathbb{C}^n} \exp(-|\partial f|^2) |det(-\partial^2 f)|^2 dvol\end{aligned}$$
*is given by the Milnor number of $f$. This is a generalization of a result in [33], see the last section for more discussion.*

### 1.3.3 LG/CY Correspondence for Weil-Peterson-type Metric and $tt^*$ Structures

Let $f:\mathbb{C}^n\to\mathbb{C}$ be a quasi-homogeneous polynomial, $F(z,u)$ be a marginal deformation of $f$:

$$F(x,u)=f(x)+\sum_{i=1}^{\nu}u^i\psi_i(x),$$

i.e. $l(\psi_i dz_1...dz_n)=l(f dz_1...dz_n)$, $i=1,\dots,\nu$ (Recall Definition 1.2.12 for the definition of $l$). We denote $M$ by the space of parameters $u^i$, which should be a small neighborhood of the origin in $\mathbb{C}^\nu$.

For each $A_a=\phi_a dz_1...dz_n$, there exists a harmonic form $w_a(u)\in\ker(\Delta_{F(\cdot,u)})$ (c.f. Proposition 6.1.2), s.t.

$$w_a(u)=A_a+\bar\partial_{F(\cdot,u)}\nu_a(u)$$

for some $\nu_a$, such that $\eta_a$ has at most polynomial growth.

**Theorem 1.3.8.** *$M$ carries a Weil-Peterson type metric $G$. More explicitly, one has*

$$G_{i\bar j}=\partial_i\bar\partial_j\log h(w_0(u),w_0(u))=\frac{h(C_iw_0(u),C_jw_0(u))}{h(w_0(u),w_0(u))},$$

*where $G_{i\bar j}=G(\partial_i,J\bar\partial_j)$,$\partial_i:=\frac{\partial}{\partial u_i},\bar\partial_j:=\frac{\partial}{\partial\bar u_j}$, $J$ is the canonical complex structure on $\mathbb{C}^n$.*

**Now assume that $f$ is homogeneous of degree $n$.** Fix a homogeneous basis $\{\phi_a\}_{a=0}^{\mu-1}$ of $\mathrm{Jac}(f)$, such that $l(A_a)\le l(A_b)$ if $a<b$, $\phi_0=1$. Moreover, assume that $\{\phi_{a_i}\}_{i=0}^{\mu'-1}$ is a basis of $\mathrm{Jac}(f)'$. Let $|u|$ be small enough, such that $\{\phi_a\}_{a=0}^{\mu-1}$ is still a basis of $\mathrm{Jac}(F(\cdot,u))$, $\{\gamma_k\}_{k=0}^{\mu}$ constructed in Section 1.2.3 is still a basis of $H_n(\mathbb{C}^n,F(\cdot,u)^{-\infty})$.

By [25] or Theorem 1.3.3, one can see that $\{e^{F+\bar F}w_a\}$ ($\{e^{-F-\bar F}*w_a\}$) is a basis of

$H^n(\mathbb{C}^n, F(\cdot, u)^{-\infty})$ $(H^n(\mathbb{C}^n, F(\cdot, u)^{+\infty}))$. Consequently, the matrix

$$(\mathcal{A})_{ka} := \int_{\gamma_k} e^{F+\bar{F}} w_a$$

and

$$(\tilde{\mathcal{A}})_{ka} := \int_{\tilde{\gamma}_k} e^{-F-\bar{F}} * w_a$$

are invertible.

Let

$$(\mathcal{B})_{ka} := \int_{\gamma_k} e^F A_a,$$

and

$$(\tilde{\mathcal{B}})_{ka} := \int_{\gamma_k} e^{-F} * A_a,$$

then there exist matrices $\mathcal{T}$ and $\tilde{\mathcal{T}}$, such that $\mathcal{B} = \mathcal{T}\mathcal{A}$ and $\tilde{\mathcal{B}} = \tilde{\mathcal{T}}\tilde{\mathcal{A}}$.

In [11], using Leznov-Saveliev method, Cecotti-Vafa shows that there exists (c.f. (A.9) in [11]) a block-diagonal and holomorphic matrix $\mathcal{F}$, a unit lower triangular matrix $\mathcal{N}$, such that $\mathcal{T} = e^{\mathcal{F}}\mathcal{N}$. As a result, there exists a holomorphic function $\lambda(u)$, such that

$$\int_{\gamma_k} e^{F+\bar{F}} w_0 = e^{\lambda(u)} \int_{\gamma_k} e^F dz_1 ... dz_n.$$

However, we have a more explicit description of $\mathcal{T}$. Moreover, $\lambda(u) \equiv 0$ :

**Theorem 1.3.9.** *$\mathcal{T}$ and $\tilde{\mathcal{T}}$ are (block) unit lower triangular matrix. More explicitly,*

$$(\mathcal{T})_{aa} = 1 \text{ for } 0 \le a \le \mu - 1;$$

$$(\mathcal{T})_{ab} = 0 \text{ if } a < b;$$

$$(\mathcal{T})_{ab} = \begin{cases} 0, \text{ if } \frac{l(A_a)-l(A_b)}{n} \notin \mathbb{Z}^+; \\ \frac{(-1)^{l_{ab}}}{l_{ab}!} \sum_{l_c = l_b} \int_{\mathbb{C}^n} \bar{f}^{l_{ab}} A_a \wedge *\bar{w}_c h^{cb}, \text{ if } l_{ab} := \frac{l(A_a)-l(A_b)}{n} \in \mathbb{Z}^+. \end{cases}$$

*(We will see that by Agmon estimate, $\int_{\mathbb{C}^n} \bar{f}^{l_{ab}} A_a \wedge *\bar{w}_b < \infty$. )*

*Here $(h^{ab})$ is the inverse of $(h_{ab})$, $h_{ab} = h(w_a, w_b)$. In particular,*

$$\int_{\gamma_k} e^{F+\bar{F}} w_0 = \int_{\gamma_k} e^F dz_1 ... dz_n,$$

$$\int_{\tilde{\gamma}_k} e^{-F-\bar{F}} * w_0 = \int_{\tilde{\gamma}_k} e^{-F} * dz_1 ... dz_n,$$

Let $X_u = \{[z] \in \mathbb{C}P^{n-1} : F(z,u) = 0\}$. In Section 6.3, we will define a map $R : \mathrm{Jac}(F(\cdot,u))' \to H^{n-2}(X_u)_0[-1]$ that preserves the Hodge filtration. In particular, $R(1)$ is a nowhere vanishing holomorphic $(n-2)$-form on $X_u$. Now let $G^{CY} := \partial\bar{\partial} \log \int_{X_u} R(1) \wedge *\overline{R(1)}$, then $G^{CY}$ is the Weil-Peterson metric on $M$. It follows from Theorem 1.3.9, that

**Theorem 1.3.10.** $G = G^{CY}$.

Eventually,

**Theorem 1.3.11.** *The small $tt^*$ structure on LG's side (See Definition 1.2.14) and the $tt^*$ structure on CY's side (See Definition 2.4.7) are isomorphic.*

# Chapter 2

# Preliminary

## 2.1 Witten Deformation on Closed Manifolds

In this section, we will have a brief review on Witten deformation on closed manifolds.

### 2.1.1 Hodge Theory

Let $(M, g)$ be an $n$-dimensional closed Riemannian manifold.

Recall that with the exterior differential $d : \Omega^k(M) \to \Omega^{k+1}(M)$, we have a cochain complex

$$(\Omega^*(M), d) : 0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \xrightarrow{d} 0$$

called the de Rham complex, with de Rham cohomology:

$$H^k_{\mathrm{dR}}(M; \mathbb{R}) := \frac{\ker d|_{\Omega^k(M)}}{\operatorname{Im} d|_{\Omega^{k-1}(M)}}.$$

The Riemannian metric $g$ induces an inner-product on $\Omega^k(M)$ via

$$(\alpha, \beta)_{L^2} = \int_M \alpha \wedge *\beta$$

where $* : \Omega^k(M) \to \Omega^{n-k}(M)$ is the Hodge star operator. If we denote by $\delta : \Omega^k(M) \to \Omega^{k-1}(M)$ by

$$\delta = (-1)^{nk+n+1} * d*,$$

it is easy to see that

$$\langle d\alpha, \beta \rangle = \langle \alpha, \delta\beta \rangle .$$

That is, $\delta$ if the formally adjoint of $d$.

If we denote the de Rham-Hodge operator by $D = d + \delta$, then the Hodge Laplacian $\Delta := D^2 = d\delta + \delta d$. In particular, one has the Kodaira-Hodge decomposition theorem holds,

$$\Omega^*(M) = \ker \Delta \oplus \operatorname{Im} d \oplus \operatorname{Im} \delta.$$

We call $\alpha$ is a harmonic form if $\alpha \in \ker(\Delta)$. As a corollary, we have the isomorphism

$$\ker \Delta \simeq H^*_{dR}(M; \mathbb{R}).$$

We can obtain a nice local expression if we introduce the Clifford operators, which give an action of $TM$ on $\Omega^*(M)$. If $v \in TM$, set

$$c(v) := v^\# \wedge -\iota_v \quad \text{and} \quad \hat{c}(v) := v^\# \wedge +\iota_v$$

where $\iota_v$ is the interior product and $v^\# \in T^*M$ is the dual of $v$ under $g$. We will also denote $c\left(v^{\#}\right) = c(v)$ and $\hat{c}\left(v^{\#}\right) = \hat{c}(v)$. One has Clifford relations for these maps:

$$\begin{aligned} c(v)c(w) + c(w)c(v) &= -2g(v,w) \\ \hat{c}(v)\hat{c}(w) + \hat{c}(w)\hat{c}(v) &= 2g(v,w) \\ c(v)\hat{c}(w) + \hat{c}(w)c(v) &= 0 \end{aligned}$$

Given a local orthonormal frame $\{e_i\}_{i=1}^n$ for $TM$ with corresponding dual basis $\{e^i\}_{i=1}^n$ of $T^*M$, and denoting by $\nabla$ the Levi-Civita connection, we have the following local expressions:

$$\begin{aligned} d &= \sum_{i=1}^n e^i \wedge \nabla_{e_i}, \\ \delta &= -\sum_{i=1}^n \iota_{e_i} \nabla_{e_i}, \\ D &= \sum_{i=1}^n c\left(e_i\right) \nabla_{e_i}. \end{aligned}$$

### 2.1.2 Witten Deformation

Let $f \in C^\infty(M)$ be a Morse function on $M$, and for $T \in \mathbb{R}$ define the deformation of the exterior diferential differential as:

$$d_{Tf} = e^{-Tf} d e^{Tf} = d + T df \wedge .$$

One can see that $d_{Tf} : \Omega^k(M) \to \Omega^{k+1}(M)$, for any $k$. Also,

$$d_{Tf}^2 = e^{-Tf} d^2 e^{Tf} = 0.$$

Therefore, we have a cochain complex

$$(\Omega^\bullet(M), d_{Tf}) : 0 \to \Omega^0(M) \xrightarrow{d_{Tf}} \Omega^1(M) \xrightarrow{d_{Tf}} \cdots \xrightarrow{d_{Tf}} \Omega^n(M) \xrightarrow{d_{Tf}} 0$$

and the cohomology space

$$H^*_{Tf,\mathrm{dR}}(M;\mathbb{R}) := \frac{\ker d_{Tf}|_{\Omega^*(M)}}{\operatorname{Im} d_{Tf}|_{\Omega^{*-1}(M)}}$$

Let $\delta_{Tf}$ be the formal adjoint of $d_{Tf}$ with respect to $L^2$ metric

$$(\alpha,\beta)_{L^2} = \int_M \alpha \wedge *\beta.$$

That is,

$$\langle d_{Tf}\alpha, \beta\rangle = \langle \alpha, \delta_{Tf}\beta\rangle .$$

If we denote the de Rham-Hodge operator by $D_T = d_{Tf} + \delta_{Tf}$, then the Hodge Laplacian $\Delta_T := D_T^2 = d_{Tf}\delta_{Tf} + \delta_{Tf}d_{Tf}$. In particular, one has the Kodaira-Hodge decomposition theorem holds,

$$\Omega^*(M) = \ker \Delta_T \oplus \operatorname{Im} d_{Tf} \oplus \operatorname{Im} \delta_{Tf}.$$

We call $\alpha$ is a harmonic form if $\alpha \in \ker(\Delta_T)$. As a corollary, we have the isomorphism

$$\ker \Delta_T \simeq H^*_{Tf,dR}(M;\mathbb{R}).$$

We can obtain a nice local expression by using the Clifford operators: Given a local orthonormal frame $\{e_i\}_{i=1}^n$ of $TM$ with corresponding dual basis $\{e^i\}_{i=1}^n$ of $T^*M$, and denoting by $\nabla$ the Levi-Civita connection, we have the following local expressions:

$$d_{Tf} = \sum_{i=1}^{n} e^i \wedge \nabla_{e_i} + T df \wedge,$$
$$\delta = -\sum_{i=1}^{n} \iota_{e_i} \nabla_{e_i} + T \iota_{\nabla f},$$
$$D = \sum_{i=1}^{n} c\left(e_i\right) \nabla_{e_i} + T\hat{c}(df).$$

Moreover, by a direct computation, $\Delta_T = \Delta + T\nabla^2_{e_i,e_j} f c(e_i)\hat{c}(e_j) + T^2|\nabla f|^2$.

### 2.1.3 Witten's Idea

Now, we can briefly illustrate Witten's idea here: look at an eigenform $u$ of $\Delta_T$ with respect to eigenvalue $\lambda$, one can see easily that if the value of $u$ doesn't concentrate on the nearby of cirtical points, i.e., the place where $\nabla f = 0$, then it must concentrate at somewhere, say $U$, that $|\nabla f| > c$ for some $c > 0$. Restricted on $U$, $\Delta_T \geq \frac{cT^2}{2}$ if $T$ is big. As a result, one must have $\lambda \geq \frac{cT^2}{2}$. In other words, if we focus on the eigenspaces with respect to eigenvalues $\lambda = O(1)$ as $T \to \infty$, then we can localize our discussion at the critical points of $f$. See [27, 25] or Chapter 3 for more details.

## 2.2 Analytic Torsion on Closed Manifolds

In this section, we will review analytic torsion on closed manifolds.

### 2.2.1 The Finite-dimensional Case

If $V$ is a finite-dimensional $k$-vector space of dimension $n$, then we define the determinant of $V$ by

$$\det(V) := \Lambda^n V.$$

One can see that det is a functor from the category of finite-dimensional vector space to the category of one-dimensional vector space.

If $L$ is a one-dimensional vector space, then we have a canonical isomorphism

$$\mathrm{End}(L) \cong k$$

Under this identification the morphism $\det(A) : \det V \to \det V$ induced by a morphism $A : V \to V$ is exactly mapped to the element $\det(A) \in k$.

We now consider a finite chain complex over $k$,

$$\mathcal{C} : \cdots \to C^{n-1} \to C^n \to C^{n+1} \to \ldots$$

of finite-dimensional $k$-vector spaces.

**Definition 2.2.1.** *We define the determinant of the chain complex $\mathcal{C}$ to be the one dimensional $k$-vector space*

$$\det \mathcal{C} := \bigotimes_n (\det C^n)^{(-1)^n}$$

*The determinant $\det(\mathcal{C})$ only depends on the underlying $\mathbb{Z}$-graded vector space of $\mathcal{C}$ and not on the differential.*

The cohomology $H(\mathcal{C})$ of the chain complex $\mathcal{C}$ can be thought as a chain complex with trivial differentials. Hence $\det H(\mathcal{C})$ is also well-defined.

Moreover, one has

**Proposition 2.2.2.** *We have the so-called torsion isomorphism*

$$\tau_{\mathcal{C}} : \det \mathcal{C} \xrightarrow{\cong} \det H(\mathcal{C}).$$

We now assume that $k = \mathbb{R}$ or $k = \mathbb{C}$. Given a metric $h^V$ on $V$. It induces a metric $h^{\det V}$ on $\det V$.

On a chain complex $\mathcal{C}$, Let $h^{\mathcal{C}}$ be a collections of metrics $\left(h^{C_n}\right)_{n.}$ Such a metric induces a metric $h^{\det \mathcal{C}}$ on $\det \mathcal{C}$. Hence induce a metric $\tau_{\mathcal{C},*} h^{\mathcal{C}}$ on $\det H(\mathcal{C})$.

**Definition 2.2.3.** *Let $\mathcal{C}$ be a finite chain complex and $h^{\mathcal{C}}$ and $h^{H(\mathcal{C})}$ metrics on $\mathcal{C}$ and its cohomology $H(\mathcal{C})$. Then the analytic torsion*

$$T\left(\mathcal{C}, h^{\mathcal{C}}, h^{H(\mathcal{C})}\right) \in \mathbb{R}^+$$

*is defined by the relation*

$$h^{\det H(\mathcal{C})} = T\left(\mathcal{C}, h^{\mathcal{C}}, h^{H(\mathcal{C})}\right) \tau_{\mathcal{C},*} h^{\det \mathcal{C}},$$

*where $h^{\det H(\mathcal{C})}$ is the metric on $\det H(\mathcal{C})$ induced by $h^{H(\mathcal{C})}$, $h^{\det(\mathcal{C})}$ is the metric on $\det(\mathcal{C})$ induced by $h^{\mathcal{C}}$.*

We consider the $\mathbb{Z}$-graded vector space

$$C := \bigoplus_{n \in \mathbb{Z}} C^n$$

and the differential $d : C \to C$ as a linear map of degree one. The metric $h^{\mathcal{C}}$ induces a metric $h^C$ on $C$. Using $h^C$ we can define the adjoint $d^* : C \to C$ which has degree $-1$. We define the Hodge Laplace

$$\Delta := (d + d^*)^2 = dd^* + d^*d.$$

We have the Hodge decomposition

$$C \cong \ker \Delta \oplus \mathrm{Im}(d) \oplus \mathrm{Im}\left(d^*\right), \quad \mathrm{Ker}(d) = \mathrm{Ker}(d) \oplus \mathrm{Im}\,\Delta.$$

In particular, we get an isomorphism of graded vector spaces

$$H(\mathcal{C}) \cong \ker(\Delta)$$

This isomorphism induces the Hodge metric $h^{H(\mathcal{C})}_{\mathrm{Hodge}}$ on $H(\mathcal{C})$.

One can show that

**Proposition 2.2.4.** *We have the equality*

$$T\left(\mathcal{C}, h^{\mathcal{C}}, h^{H(\mathcal{C})}_{Hodge}\right) = \sqrt{\prod_{n\in\mathbb{Z}} \det\left(\Delta'_n\right)^{(-1)^n n}}$$

This proposition enables us to extend the definition of Analytic torsion to infinite-dimensional vector spaces.

### 2.2.2 Determinant in Infinite Dimensional Case

By Proposition 2.2.4, to define analytic torsion, we have to extend the definition of the determinant of an operator to the infinite-dimensional case.

First, observe that if $V$ is a finite dimensional vector space of dimension $n$ over $\mathbb{R}$ or $\mathbb{C}$, $A : V \to V$ is positive definite, then $A$ is diagonalizable. Suppose $A$ has eigenvalues $\lambda_1, ...\lambda_n$ (Counting with multiplicity), then

$$\ln\det(A) = \sum_{i=1}^{n} \ln(\lambda_i) = \sum_{i=1}^{n} \left.\frac{\partial \lambda_i^s}{\partial s}\right|_{s=0}.$$

Now define $Z_A(s) := \sum_{i=1}^n \lambda_i^s$. Since for $\lambda > 0$, $\lambda^s = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-t\lambda} dt$, one has

$$Z_A(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \sum_{i=1}^n e^{-t\lambda_i} dt = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \mathrm{Tr}(e^{-tA}) dt.$$

Hence, if $A = \Delta$, then we could make sense of $\det(\Delta)$ by using heat kernels.

### 2.2.3 Infinite Dimensional Case

Let now $\left(M, g^{TM}\right)$ be a closed Riemannian manifold, and let $\left(E, \nabla^E, h^E\right)$ be a flat vector bundle with metric $h^E$. Then we can equip the de Rham complex $\Omega^*(M; E)$ with a a metric $h_{L^2}^{\Omega(M;E)}$ given by

$$h_{L^2}^{\Omega(M;E)}(\alpha, \beta) = \int_M (\alpha, \beta)_{g^{TM}, h^E} \mathrm{dvol}_M,$$

where $(\alpha, \beta)_{g^{TM}, h^E}$ is defined as follows: if $\alpha = \omega \otimes s, \beta = \omega' \otimes s' \in \Omega^*(M, E)$ for $\omega, \omega' \in \Omega^*(M), s, s' \in \Gamma(M, E)$, then

$$(\alpha, \beta)_{g^{TM}, h^E} := g^{TM}(\omega, \omega') h^E(s, s').$$

Now the connection $\nabla^E : \Omega^0(M, E) \to \Omega^1(M, E)$ extends uniquely to a derivation of $\Omega(M)$-modules

$$d^E : \Omega^*(M, E) \to \Omega^{*+1}(M, E)$$

of degree one and $(d^E)^2 = 0$. Then the Laplace operator

$$\Delta := \left(d^{E,*} + d^E\right)^2$$

preserves degree and its components

$$\Delta_k : \Omega^k(M,E) \to \Omega^k(M,E), \quad k \in \mathbb{N}$$

are of Laplace type.

Notice that on a closed manifold, the heat kernel $K(t,x,x)$ of a Laplacian type operator $\Delta$ has the following asymptotic expansion near the diagonal:

$$K(t,x,y) = \sum_{k=0}^{\infty} \mathcal{E}_0(t,x,y) t^k \theta_k(x,y),$$

where $\mathcal{E}_0(t,x,y) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{d^2(x,y)}{4t}}$.

In particular, one has

$$K(t,x,x) = \sum_{k=0}^{\infty} \frac{t^k}{\sqrt{4\pi}} \theta_k(x,x),$$

and

$$\mathrm{Tr}(e^{-t\Delta}) = \sum_{k=0}^{\infty} \frac{t^{-\frac{n}{2}+k}}{\sqrt{4\pi}} \int_M \theta_k(x,x) \mathrm{dvol}_M = \sum_{k=0}^{\infty} a_k t^{-\frac{n}{2}+k}. \tag{2.1}$$

Naively, the zeta function is defined by

$$\zeta_\Delta(s) := \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \mathrm{Tr}(e^{-t\Delta}) dt.$$

However, it doesn't make sense. Hence we need to regularize it:

1. If $\ker(\Delta) \neq 0$, then for any $s \in \mathbb{C}$, $|\int_0^\infty t^{s-1}\mathrm{Tr}(e^{-t\Delta})dt| \geq \int_0^\infty t^{\mathrm{Re}(s)-1} dt = \infty$. Hence, we restrict on the orthogonal complement of $\ker(\Delta)$. Let $\Delta' = \Delta|_{\ker \Delta^\perp}$.

2. By Weyl's law, since the eigenvalues of $\Delta$ has polynomial growth, $\int_1^\infty t^{s-1}\mathrm{Tr}(e^{-t\Delta'})dt$ is holomorphic for any $s \in \mathbb{C}$.

3. By (2.1), $\frac{1}{\Gamma(s)}\int_0^1 t^{s-1}\left(\mathrm{Tr}(e^{-t\Delta'})-\sum_{k=0}^{l} t^{-\frac{n}{2}+k}a_k\right)dt$ is holomorphic for $\mathrm{Re}(s)\geq l-\frac{n}{2}$.

4. When $\mathrm{Re}(s)>>0$,

$$\frac{1}{\Gamma(s)}\int_0^1 t^{s-1}\left(\sum_{k=0}^{l} t^{-\frac{n}{2}+k}a_k\right)dt=\sum_{k=0}^{l}\frac{1}{\Gamma(s)}=\sum_{k=0}^{l}\frac{1}{(-\frac{n}{2}+k+s)\Gamma(s)}.$$

As a result, $\frac{1}{\Gamma(s)}\int_0^1 t^{s-1}\left(\sum_{k=0}^{l} t^{-\frac{n}{2}+k}a_k\right)dt$ could be extended to a holomorphic function near $s=0$.

In a word, there is a way to regularize the zeta function $\zeta_{\Delta'}(s)$ such that $\zeta_{\Delta'}(s)$ is holomorphic near $s=0$.

Now we can define $\det(\Delta'):=e^{\zeta'_{\Delta'}(0)}$.

**Definition 2.2.5.** *We define the analytic torsion of $\left(M,\nabla,h^{TM},h^E\right)$ by*

$$T_{an}\left(M,\nabla^E,h^{TM},h^E\right):=\sqrt{\prod_{k\in\mathbb{N}}(\det\Delta'_k)^{(-1)^k k}}$$

*It is the analog of $T\left(\Omega^*(M,E),h_{L^2}^{\Omega^*(M,E)},h_{Hodge}^{H^*(M,E)}\right)$.*

## 2.3 Cheeger-Muller/Bismut-Zhang Theorem

In this section, we will briefly introduce Cheeger-Muller/Bismut-Zhang theorem. For simplicity, now we let $E=M\times\mathbb{R}$ be the trivial line bundle, $(M,g)$ be a closed Riemannian manifold.

### 2.3.1 Thom-Smale-Witten Theory

**Definition 2.3.1.** *A Morse function $f \in C^\infty(M)$ is a smooth function on $M$, such that $\nabla^2 f(x)$ is non-degenerate at every critical point $x \in M$ of $f$. The index of a critical point $i_f(x)$ is the number of negative eigenvalues of $\nabla^2 f(x)$.*

*For $i \in \mathbb{N}$, we let $\mathrm{Crit}_f(i)$ denote the set of critical points of index $i$ so that $\mathrm{Crit}_f = \bigcup_{i\in\mathbb{N}} \mathrm{Crit}_f(i)$ is the set of critical points of $f$.*

**Definition 2.3.2.** *The stable (unstable) manifold $W^s(x)$ ($W^u(x)$) of a critical point $x \in \mathrm{Crit}_f$ is the subset of $M$ of all points $y \in M$ such that $\lim_{t\to\infty} \Phi_t(y) = x$ ($\lim_{t\to-\infty} \Phi_t(y) = x$). Here $\Phi_t$ is the flow with respect to $-\nabla f$.*

**Definition 2.3.3.** *We say that the triple $(M, g, f)$, where $f$ is a Morse function, satisfies the Morse-Smale condition, if for every pair of critical points $x, y \in M$ the intersection of $W^u(x)$ and $W^s(y)$ is transversal.*

Fix a Morse function $f$, the set of metric $g$ that makes the triple $(M, g, f)$ satisfy the Morse-Smale condition is a dense open set in the space of Riemannian metric on $M$.

Classical Morse theory tells us that a Morse function $f$ gives a CW structure of a closed manifold, whose cellular chain complex (called Thom-Smale-Witten chain complex) could be described as follows:

First, fix orientations on unstable manifolds $W^u(x)$. The vector space of degree $i$-chains is given by

$$C_i(W^u) := \mathbb{R}\left[\mathrm{Crit}_f(i)\right]$$

The chosen orientations of the unstable manifolds induce orientations. The same choices induce coorientations of the stable manifolds. For a pair

$$x, y \in \mathrm{Crit}_f \text{ with } i_f(y) = i_f(x) + 1$$

the intersection

$$W^u(y) \cap W^s(x)$$

is transversal, consists of finite number of curves. As an intersection of an oriented and a cooriented manifold, it is oriented. We define the multiplicity of $\gamma \in W^u(y) \cap W^s(x)$

$$m(\gamma) \in \{1, -1\}$$

such that the gradient $m(\gamma)\nabla f$ is positively oriented on $\gamma$. We can now define the differential of the Morse-Smale complex by

$$\partial : C_{i+1} \to C_i, \quad y \mapsto \sum_{x \in \mathrm{Crit}_f(i)} \sum_{\gamma \in (W^u(y) \cap W^s(x))} m(\gamma)x.$$

### 2.3.2 Milnor Metric and Ray-Singer Metric

Let $(C^*(W^u), \tilde{\partial}')$ denote the Thom-Smale-Witten cochain complex, $H^*(C^\bullet(W^u), \tilde{\partial}')$ denote the Thom-Smale-Witten cohomology. We also introduce an inner product on $C_i(W^u)$ as follows:

if $x, y \in Crit_f(i)$, then $\langle x, y\rangle = 0$ if $x \neq y$; $\langle x, x\rangle = 1$.

This induces a metric $h^{C^*(W^u)}$ on $C^*(W^u)$.

Recall the canonical map $\tau_{C^*}$ given in Proposition 2.2.2.

**Definition 2.3.4.** *We define the Milnor metric on* $\det H^*(C^\bullet(W^u), \tilde{\partial}')$ *by*

$$\| \cdot \|^{\mathcal{M}, \nabla f}_{\det H^\bullet(M, \mathbb{R})} = \tau_{C^*(W^u)} h^{\det C^*(W^u)}.$$

We now consider the de Rham complex $\Omega(M, \mathbb{R})$. Its cohomology will be denoted by $H(M, \mathbb{R})$. The Riemannian metric on $M$ induces an $L^2$- metric $h^{H(M,\mathbb{R})}_{Hodge}$ on the cohomology. We get an induced metric $h_{Hodge}^{\det H(M,\mathbb{R})}$ on $\det H(M,\mathbb{R})$ and the analytic torsion $T_{an}\left(M,d,g^{TM}\right)$, see Definition 2.2.5 (Here $d$ is the de Rham differential).

**Definition 2.3.5.** *We define the Ray-Singer metric*

$$\|\cdot\|^{RS}_{\det H^{\bullet}(M,\mathbb{R})} := T_{an}\left(M,d,g^{TM}\right)^{-1} h_{Hodge}^{\det H(M,\mathbb{R})}$$

**Theorem 2.3.1** (Cheeger-Müller/Bismut-Zhang theorem)**.** *If $\left(M,g^{TM}\right)$ is a closed odd-dimensional Riemannian manifold, then we have the equality of metrics*

$$\|\cdot\|^{RS}_{\det H^{\bullet}(M,\mathbb{R})} = \|\cdot\|^{\mathcal{M},\nabla f}_{\det H^{\bullet}(M,\mathbb{R})}.$$

This is not the most general version, we refer to [20] for the general case.

*Sketch of the proof.* If we replace the de Rham differential $d$ by Witten deformation $d+Tdf$ for $T\in\mathbb{R}$. Its cohomology will be denoted by $H(M,\mathbb{R})_T$. The Riemannian metric on $M$ induces an $L^2$- metric $h_{Hodge}^{H(M,\mathbb{R})}(T)$ on the cohomology. We get an induced metric $h_{Hodge}^{\det H(M,\mathbb{R})}(T)$ on $\det H(M,\mathbb{R})_T$ and the analytic torsion $\|\cdot\|^{\mathcal{M},\nabla f}_{\det H^{\bullet}(M,\mathbb{R})}$.

We could also define the Ray-Singer metric

$$\|\cdot\|^{RS}_{\det H^{\bullet}(M,\mathbb{R})}(T) := T_{an}\left(M,d_{Tf},g^{TM}\right)^{-1} h_{Hodge}^{\det H(M,\mathbb{R})}(T).$$

One can show that if $\dim_{\mathbb{R}} M$ is odd, $\|\cdot\|^{RS}_{\det H^{\bullet}(M,\mathbb{R})}(T)$ is independent of $T$. Let $T\to 0$, $\|\cdot\|^{RS}_{\det H^{\bullet}(M,\mathbb{R})}(T) \to \|\cdot\|^{RS}_{\det H^{\bullet}(M,\mathbb{R})}$. Let $T\to\infty$, the discussion localize at the critical points of $f$ (See our discussion in Subsection 2.1.3), eventually $\|\cdot\|^{RS}_{\det H^{\bullet}(M,\mathbb{R})}(T) \to \|\cdot\|^{\mathcal{M},\nabla f}_{\det H^{\bullet}(M,\mathbb{R})}$, which finishes the proof of Theorem 2.3.1. $\square$

## 2.4 $tt^*$ Structure on Calabi-Yau Manifolds

In this section, we will review variation of Hodge structures and $tt^*$ structures on a Calabi-Yau manifold $X$.

### 2.4.1 Variation of Hodge Structures

**Definition 2.4.1.** *A Hodge structure of weight $m$ is given by a data $(H_{\mathbb{Q}}, F^p)$ where $H_{\mathbb{Q}}$ is a finitely generated $\mathbb{Q}$-vector space, and $F^p, p = (0, \ldots, m)$ is a decreasing filtration on the complexification $H = H_{\mathbb{Q}} \otimes \mathbb{C}$ such that $F^p \oplus \overline{F^{m-p+1}} \cong H$, for all $p$. Setting $H^{p,q} = F^p \cap \bar{F}^q$, one has*

$$H = \bigoplus_{p+q=m} H^{p,q}, \quad H^{p,q} = \bar{H}^{q,p}$$

*Hence $F^p = \oplus_{p' \geq p} H^{p',m-p'}$.*

**Example 2.4.2.** *Let $X$ be a compact Kähler manifold, then*

$$\forall k \in \mathbb{N}, H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X), \quad H^{q,p} = \overline{H^{p,q}}$$

*gives a Hodge structure.*

**Definition 2.4.3.** *A polarized Hodge structure of weight $m$ is given by the data $(H_{\mathbb{Q}}, F^p, Q)$ where $(H_{\mathbb{Q}}, F^p)$ is a Hodge structure of weight $m$ and*

$$Q : H_{\mathbb{Q}} \otimes H_{\mathbb{Q}} \to \mathbb{Q}$$

*is a bilinear form satisfying the conditions*

- $Q(u, v) = (-1)^m Q(v, u)$

- $Q\left(F^p, F^{m-p+1}\right) = 0$

- *The Hermitian form* $i^{p-q}Q(u, \bar{v})$ *is positive definite , where* $u, v \in H^{p,q}$.

**Example 2.4.4.** *Let* $X$ *be a compact Kähler manifold of complex dimension* $n$*, then* $H^n(X)$ *admits a polarized Hodge structure by setting* $Q(u, v) = (-1)^{n(n-1)/2} \int_X u \wedge v$.

**Definition 2.4.5.** *A variation of polarized Hodge structure is given by the data*

$$\left(S, \mathcal{H}_{\mathbb{Q}}, \mathcal{F}^p, \nabla + \bar{\nabla}, Q\right),$$

*where*

- $S$ *is a smooth complex algebraic variety.*

- $\mathcal{H}_{\mathbb{Q}}$ *is a local system of* $\mathbb{Q}$*-vector spaces on* $S$.

- $\mathcal{H} = \mathcal{H}_{\mathbb{Q}} \otimes \mathcal{O}_S$ *is a holomorphic vector bundle with a filtration* $\mathcal{F}^p$ *by holomorphic sub-bundles.*

- $\nabla + \bar{\nabla} : \mathcal{H} \to \Omega^1_S \otimes \mathcal{H}$ *is an integrable connection. Here* $\nabla$ *is the* $(1, 0)$ *component of the connection and* $\bar{\nabla}$ *the* $(0, 1)$ *component.*

- $(\nabla + \bar{\nabla})\mathcal{H}_{\mathbb{Q}} = 0$

- *For each* $s \in S$*, on each fiber the induced data* $(\mathcal{H}_{\mathbb{Q},s}, \mathcal{F}^p_s)$ *gives a Hodge structure of weight* $m$.

- *The Griffiths transversality conditions*

$$(\nabla + \bar{\nabla})\mathcal{F}^p \subseteq \Omega^1_S \otimes \mathcal{F}^{p-1}$$

  *are satisfied.*

- *The bilinear form $Q$*

$$Q : \mathcal{H}_{\mathbb{Q}} \otimes \mathcal{H}_{\mathbb{Q}} \to \mathbb{Q}$$

*satisfying $(\nabla + \bar{\nabla})Q = 0$ and inducing polarized Hodge structure on each fiber.*

**Example 2.4.6.** *Let $\pi : \mathcal{X} \to S$ be a family of compact Kähler manifold with typical fiber $X$ ($\dim_{\mathbb{C}} X = n$). Consider vector bundle $\mathcal{H}_{\mathbb{Q}} = R^n\pi_*\mathbb{Q}$. Fiberwisely, $\mathcal{H}_{\mathbb{Q}}$ is the space of middle cohomology of the fiber. Let $\mathcal{F}$ be the filtration induced by fiberwise filtration, $\nabla + \bar{\nabla}$ be the Gauss-Manin connection on $\mathcal{H}_{\mathbb{Q}}$, $Q$ be the fiberwise pairing. Then $(S, \mathcal{H}_{\mathbb{Q}}, \mathcal{F}, \nabla, Q)$ gives a variation of polarized Hodge structure.*

**Variation of Polarized Hodge Structure and $tt^*$ Structure**

Consider a variation of Hodge structure $\left(S, \mathcal{H}_{\mathbb{Q}}, \mathcal{F}^p, \nabla + \bar{\nabla}, Q\right)$, and let $\mathcal{H} = \mathcal{H}_Q \otimes \mathcal{O}_S$, then $\mathcal{H}$ admits a natural real structure $\kappa$. Next, let $g(u, v) := i^{p-q}Q(u, \bar{v}), u, v \in \mathcal{H}^{p,q}$ be the hermitian metric on $\mathcal{H}$, $\{e_k\}$ be a local frame with respect to metric $h$. By Griffiths transversality condition, we could decompose $\nabla = D + C$, $\bar{\nabla} = \bar{D} + \bar{C}$, s.t. $D + \bar{D}$ is the Chern connection with respect to metric $h$. Then one can check easily that $(\mathcal{H} \to S, \kappa, g, D, \bar{D}, C, \bar{C})$ gives a $tt^*$ structure. Hence, we can view a $tt^*$ structure as a generalized version of variation of polarized Hodge structure.

### 2.4.2 $tt^*$ Structure on Calabi-Yau Manifolds

Let $\mathcal{M}$ be the moduli stack of complex structures of Calabi-Yau manifold $X$ of dimension $n$, with the universal family $\pi : \mathfrak{X} \to \mathcal{M}$. Todorov-Tian's smoothness theorem implies that $\mathcal{M}$ is smooth of dimension $\dim \mathrm{H}^1(X, T_X) = h^{n-1,1}(X)$. We denote by $\mathcal{H}_{\mathbb{Q}}^{CY}$ the vector bundle

$$\mathcal{H}_{\mathbb{Q}}^{CY} = R^n\pi_*(\mathbb{Q})_0$$

on $\mathcal{M}$. Fiberwisely, $\mathcal{H}_Q^{CY}$ is the space of primitive forms in the middle cohomology of the fiber. Let $\mathcal{H}^{CY} := \mathcal{H}_{\mathbb{Q}}^{CY} \otimes \mathcal{O}_{\mathcal{M}}$, then $\mathcal{H}^{CY}$ admits a real structure $\kappa^{CY}$. Also $\mathcal{H}^{CY}$ has a flat holomorphic structure given by the Gauss-Manin connection. We will use $\nabla^{GM}$ to denote the $(1,0)$ component of the Gauss-Manin connection and $\bar{\nabla}^{GM}$ the $(0,1)$ component. Let $F^p\mathcal{H}^{CY}$ be the Hodge filtration, and

$$\mathcal{H}^{p,n-p} := F^p\mathcal{H}^{CY}/F^{p+1}\mathcal{H}^{CY}$$

is the Hodge bundle of type $(p, n-p)$.

By Griffiths transversality condition, $\mathcal{L}^{CY} := \mathcal{H}^{n,0}$ is a holomorphic subbundle of $\mathcal{H}^{CY}$. $\mathcal{L}^{CY}$ is called the vacuum line bundle in the physics literature. For a given point $[X] \in \mathcal{M}, \mathcal{L}_{[X]}^{CY}$ is the space of holomorphic volume form on $X$. Let $Q^{CY}(u,v) := (-1)^{n(n-1)/2}\int_X u \wedge v$, and $g^{CY}(u,v) := i^{p-q}Q(u,\bar{v})$ for $u, v \in H^{p,q}(X)$, then for any holomorphic section of $\Omega \in \mathcal{L}^{CY}$, the Weil-Peterson metric $G^{CY}$ on $\mathcal{M}$ is given by $\partial^{\mathcal{M}}\bar{\partial}^{\mathcal{M}}\log(g(\Omega,\Omega))$. By our discussion in last subsection, the Gauss-Manin connection admits a decomposition $\nabla^{GM} = D^{CY} + C^{CY}$, $\bar{\nabla}^{GM} = \bar{D}^{CY} + \bar{C}^{CY}$. Moreover, in this case, $C^{CY} : \mathcal{H}^{p,n-p} \to \Omega^{(1,0)}(\mathcal{M}) \otimes \mathcal{H}^{p-1,n-p+1}$ is the Kodaira-Spencer map. Now

$$(\mathcal{H}^{CY} \to \mathcal{M}, \kappa^{CY}, g^{CY}, D^{CY}, \bar{D}^{CY}, C^{CY}, \bar{C}^{CY}) \tag{2.2}$$

is a $tt^*$ structure.

Let $f : \mathbb{C}^n \to \mathbb{C}$ be a homogeneous polynomial of degree $n$, $F(z,u)$ be a marginal deformation of $f$:

$$F(x,u) = f(x) + \sum_{i=1}^{\nu} u^i \psi_i(x),$$

i.e. $\phi_i$ has degree $n$.

We denote $M$ by the space of parameters $u^i$, which should be a small neighborhood of the origin in $\mathbb{C}^\nu$. For $u \in M$, $X_u := \{[z] \in \mathbb{C}P^{n-1} : f(z) = 0\}$ is a Calabi-Yau manifold of dimension $n-2$. There exists a map $i : M \to \mathcal{M}$.

**Definition 2.4.7.** *The $tt^*$ structure $(\mathcal{H}^{CY} \to M, \kappa^{CY}, g^{CY}, D^{CY}, \bar{D}^{CY}, C^{CY}, \bar{C}^{CY})$ on Calabi-Yau's side is the the restriction of $tt^*$ structure (2.2) on $M$.*

## 2.5 BCOV Torsion and Holomophic Anomaly Formula on Calabi-Yau Manifolds

We review the story of the Calabi-Yau B-model first: We consider a family of Calabi-Yau manifolds $\mathcal{X} \to M$ with a typical fiber $X$, where $M$ parameterize different complex structure of $X$. Indeed, it is well known that the CY B-model concerns the deformation of the complex structure. The genus 0 theory is equivalent to the variation of Hodge structure (equivalently, $tt^*$ structure). The study of the higher genus theory is much more challenging and interesting. In this direction, Bershadshy-Cecotti-Ooguri-Vafa (BCOV) showed that the genus one term $F_1$ for CY B-model admits a holomorphic anomaly equation as follows (see Subsection 2.4.2 for further explanations of each term)

$$\partial^M \bar{\partial}^M F_1 = \frac{1}{2} \operatorname{tr} C^{CY} \bar{C}^{CY} - \frac{G^{CY}}{24} \chi(X), \tag{2.3}$$

where $\chi(X)$ is the Euler number of $X$.

Let $\Delta^{p,q}$ be the restrict of Hodge Laplacian on $p, q$ forms $\Omega^{p,q}(X)$, then the BCOV torsion is defined as

$$\tau_{BCOV} = \Pi_{p,q} \det \left(\Delta^{p,q}\right)^{(-1)^{p+q} p}.$$

See Subsection 2.2.2 for the definition of $\det(\Delta)$.

For a family of Calabi-Yau B-model $\mathcal{X} \to M$, fixing a Kähler structure on $X$, then we define fiberwise BCOV torsion, still denoted by $\tau_{BCOV}$.

Next, condsider the determinant line bundle $\lambda = \bigwedge_{0\le p,q\le n} (\det R^q \pi_* \Omega^p(X/M))^{(-1)^{p+q}p}$, where $\Omega^p(X/M)$ is the sheaf of relative $p$-forms. Then on line bundle $\lambda \to M$, there are two natural metrics, the usual $L^2$ metric $\|\cdot\|_{L^2}$ induced by the harmonic forms, and the Quillen metric $\|\cdot\|_Q$ given by $\|\cdot\|_Q = \|\cdot\|_{L^2}\tau_{BCOV}^{-1}$.

Hence

$$\partial^M\bar{\partial}^M \log \|\cdot\|_Q = \partial^M\bar{\partial}^M \log \|\cdot\|_{L^2} - \partial^M\bar{\partial}^M \log \tau_{BCOV}.$$

Then using Bismut-Gillet-Soulé (BGS) type curvature formula [34], BCOV [3] and Fan-Fang-Yoshikawa [35] shows that

$$\partial^M\bar{\partial}^M \log \|\cdot\|_Q = -\frac{G^{CY}}{24}\chi(X),$$

$$\partial^M\bar{\partial}^M \log \|\cdot\|_{L^2} = \frac{1}{2}\operatorname{tr} C^{CY}\bar{C}^{CY}$$

Thus, comparing with (2.3), the $F_1$ term should be $\log \tau_{BCOV}$, up to some holomorphic and anti-holomorphic corrections.

## 2.6 Motivation of This Thesis Project

We can now discuss why this thesis project is significant to us in greater detail. The CY B-model is concerned with the deformation of the complex structure, as discussed in Section 2.5: The genus-zero theory is equivalent to the study of $tt^*$ structure (see Definition 2.4.7) of Calabi-Yau manifolds; the genus-one term $F_1$ is computed by BCOV torsion.

Then in keeping with the spirit of the LG/CY correspondence, we should have a similar story for the LG B-model, which involves the deformation of singularities. Indeed, its genus 0 theory is given by Saito's theory of primitive forms and higher residue pairing(c.f. [8, 9]), originating in Saito's study of period integrals over vanishing cycles associated to an isolated singularity. Compared with the CY B-model, one might conjecture that the genus one term could be expressed as a torsion type invariant (We call it BCOV type torsion for LG B-model). In [33] and [36], Fan-Fang and Shen-Xu-Yu defined such a counterpart of the BCOV torsion for LG models on $\mathbb{C}^n$. Moreover, in [37], X. Tang deduced a holomorphic anomaly formula for this torsion for the case of $\mathbb{C}^n$.

It will be interesting to ask if the LG/CY correspondence holds for the B-model. To address this question for the genus 0 case, we show LG/CY correspondence for $tt*$ structures in Chapter 6. In fact, Fan-Lan-Yang in [23] partially prove that the two $tt^*$ structures are isomorphic via the CY/LG correspondence, except for the real structures . Here we use different methods to show the full LG/CY correspondence: we introduce two $U(1)$ actions that act as certain bi-grading for LG B-models. With the help of these two $U(1)$ actions, we also show LG/CY correspondence for Weil-Peterson type metrics. Additionly, in our method, the Agmon estimate derived in Chapter 3 plays an essential role.

Following that, we investigate the LG/CY correspondence for the genus-one term in B-models.The genus-one term of the LG B-model is conjectured to be computed via some analytic torsion with respect to the Witten deformation $\bar{\partial}_f = \bar{\partial} + \partial f \wedge$. Hence, the first section of this thesis studies Witten deformations on noncompact manifolds. The Agmon estimate in Chapter 3 also plays an important part in the heat kernel expansion and the proof of LG/CY correspondence for genus 0 theory. As discussed in Subsection 2.2.2, to make sense of the determinant of Laplacian, i.e. analytic torsion, one must first understand the heat kernel expansion, which is our main task in Chapter 4. In Chapter 5, we define analytic torsion for Witten deformations on noncompact manifolds using the heat kernel expansion from Chapter 4. To study LG/CY correspondence for genus

1 term, we must generalize the previous discussion to the case of families of LG models and the complex/holomorphic setting, which is still under exploration.

# Part I

# Analysis of Witten Deformation on Noncompact Manifolds

# Chapter 3

# On Agmon Estimate and Thom-Smale-Witten Complex

In this chapter, we study the Agmon estimate, and explore the relations between the Thom-Smale complex for a Morse function $f$ on a noncompact manifold $M$ and the deformed de Rham complex with respect to $f$.

The crucial technical part of this work is the Agmon estimate for eigenforms of the Witten Laplacian which is essential in extending the usual analysis from compact setting to the noncompact case. The Agmon estimate was discovered by S. Agmon in his study of $N$-body Schrödinger operators in the Euclidean setting and has found many important applications. The exponential decay of the eigenfunction is expressed in terms of the so-called Agmon distance, Cf. [32]. We make essential use of this Agmon estimate to carry out the isomorphism between the Witten instanton complex defined in terms of eigenspaces corresponding to the small eigenvalues with the Thom-Smale complex defined in terms of the critical point data of the function. We remark that the Agmon estimate near the critical points also plays an important role in the compact case , see [38] and also [20]. The novelty here is that we make essential use of the exponential decay at spatial infinity provided by the Agmon estimate. Later on, we will see that Agmon estimate also plays a very important part in the proof of LG/CY correspondence for Weil-Peterson metrics.

The first difficulty one encounters here is the presence of continuous spectrum on a noncompact manifolds and for that one has to impose certain tameness conditions. This consists of the bounded geometry requirement for the manifold as well as growth conditions for the function. The notion of strong tameness is introduced in [14] in the Kähler setting which guarantees the discreteness of the spectrum for the Witten Laplacian. Here we introduce a slightly weaker notion which allows continuous spectrum but only outside a large interval starting from 0.

It is important to note that, and this is another new phenomenon in the noncompact case, the Thom-Smale complex may not be a complex in general. Namely, the square of its boundary operator need not be zero, since $M$ is noncompact. However we prove that with the tameness condition, it is.

## 3.1 The Spectrum of Witten Laplacian

In this section we study the spectral theory of the Witten Laplacian on noncompact manifolds. In particular we establish the Kodaira decomposition and the Hodge theorem for the Witten Laplacian under our tameness condition.

### 3.1.1 Essential Self-adjointness of $d_f + \delta_f$

**Theorem 3.1.1.** *On a complete Riemannian manifold, if*

$$\limsup_{p\to\infty} \frac{|\nabla^2 f|(p)}{|\nabla f|^2(p)} < \infty,$$

*then $d_f + \delta_f$ is essentially self-adjoint.*

*Proof.* Since $\limsup_{p\to\infty} \frac{|\nabla^2 f|(p)}{|\nabla f|^2(p)} < \infty$, $\Box_f$ is bounded from below by Proposition 1.2.5. The rest of the proof is essentially the same as in Section 4 of [39]; see also the proof of Theorem 1.17 in [40]. $\Box$

### 3.1.2 On the Spectrum of $\Box_{Tf}$

From now on we will assume that $(M, g, f)$ is well tame. Let $K$ be a compact subset, which can be taken to be a compact submanifold with boundaries that contains the closure of a ball of sufficiently large radius of $M$ (we will make a more specific choice of $K$ later in section 3.3), such that $\epsilon_f(K) := \inf_{M-K} |\nabla f| > 0, c_f(K) := \sup_{M-K} \frac{|\nabla^2 f|}{|\nabla f|^2} < \infty$. Then on $M - K$,

$$|\nabla f| > \frac{1}{2}\epsilon_f, \quad |\nabla^2 f| < 2c_f|\nabla f|^2. \tag{3.1}$$

Let $C_K = \max_K |\nabla^2 f|$. First, we establish the following basic lemma.

**Lemma 3.1.1.** *Fix any $b \in (0, 1)$, there exists $T_1 = T_1(c_f, C_R, \epsilon_f, b) \geq 0$ so that whenever $T > T_1$, $\phi \in \mathrm{Dom}(\Box_{Tf})$*

$$\begin{aligned} \int_M (\Box_{Tf}\phi, \phi)\mathrm{dvol} \;\; &\geq \;\; \int_M (\nabla\phi, \nabla\phi)\mathrm{dvol} + \int_{M-K} b^2T^2|\nabla f|^2(\phi, \phi)\mathrm{dvol} \\ &\quad -(C_R + TC_K)\int_K (\phi, \phi)\mathrm{dvol}. \end{aligned} \tag{3.2}$$

*Here $C_R$ is a constant depending only on the sectional curvature bounds of $g$.*

*Proof.* It suffices to show the inequality for a compactly supported smooth form. By

Proposition 1.2.5, together with the Bochner-Weitzenböck formula, we have

$$\begin{aligned}\int_M (\Box_{Tf}\phi,\phi)\mathrm{dvol} \;\; &\geq \;\; \int_M (\nabla\phi,\nabla\phi)\mathrm{dvol} - (C_R + TC_K)\int_K (\phi,\phi)\mathrm{dvol} \\ &\quad + \int_{M-K} e_T(p)(\phi,\phi)\mathrm{dvol},\end{aligned}$$

where $e_T = T^2|\nabla f|^2(1 - \frac{4c_f}{T} - \frac{4C_R}{T^2\epsilon_f^2})$. Thus, for any $b \in (0,1)$, let

$$T_1(K) := \max\{\frac{8c_f}{1-b^2}, \frac{\sqrt{8C_R}}{\epsilon_f\sqrt{1-b^2}}\}. \tag{3.3}$$

Then whenever $T > T_1$, one can see $1 - \frac{4c_f}{T} - \frac{4C_R}{T^2\epsilon_f^2} > b^2$. Consequently,

$$\begin{aligned}\int_M (\Box_{Tf}\phi,\phi)\mathrm{dvol} \;\; &\geq \;\; \int_M (\nabla\phi,\nabla\phi)\mathrm{dvol} + \int_{M-K} |bT\nabla f|^2(\phi,\phi)\mathrm{dvol} \\ &\quad -(C_R + TC_K)\int_K (\phi,\phi)\mathrm{dvol}.\end{aligned}$$

□

**Remark 3.1.2.** *When $(M,g,f)$ is strongly tame, we can take $K$ to be sufficiently "large" so that $c_f$ and $\frac{1}{\epsilon_f}$ are as small as one wants. As a result, $T_1$ can be made as small as one wants by choosing appropriate $K$.*

**Theorem 3.1.2.** *Let $\sigma$ be the set of spectrum of $\Box_{Tf}$. Then when $T > T_1$, $\sigma \cap [0, (\frac{b\epsilon_f}{2})^2T^2]$ consists of a finite number of eigenvalues of finite multiplicity.*

*Proof.* Let $P : L^2\Lambda^*(M) \to L^2\Lambda^*(M)$ be the integral of the spectral measure of $\Box_{Tf}$ on $[0, (\frac{b\epsilon_f}{2})^2T^2]$. It suffices to prove that $L := Im(P)$ is finite dimensional. For any $\phi \in L$, we have

$$\int_M (\Box_{Tf}\phi,\phi)dvol \leq (\frac{b\epsilon_f}{2})^2T^2\int_M |\phi|^2 dvol. \tag{3.4}$$

Combining with (3.2), we have

$$\begin{aligned}(\frac{b\epsilon_f}{2})^2T^2\int_M|\phi|^2dvol \quad \geq \quad & \int_M(\nabla\phi,\nabla\phi)\mathrm{dvol}+\int_{M-K}|bT\nabla f|^2(\phi,\phi)\mathrm{dvol}\\ & -(C_R+TC_K)\int_K(\phi,\phi)\mathrm{dvol}\end{aligned}$$

provided $T > T_1$. That is,

$$\begin{gathered}\int_M(\nabla\phi,\nabla\phi)\mathrm{dvol}+\int_{M-K}|bT\nabla f|^2(\phi,\phi)\mathrm{dvol}\\ \leq (\frac{b\epsilon_f}{2})^2T^2(1+\frac{4C_R}{(b\epsilon_f)^2T^2}+\frac{4C_K}{(b\epsilon_f)^2T})\int_K(\phi,\phi)\mathrm{dvol}+(\frac{b\epsilon_f}{2})^2T^2\int_{M-K}(\phi,\phi)\mathrm{dvol}.\end{gathered}$$

Since $|bT\nabla f|^2 > (\frac{b\epsilon_f}{2})^2T^2$ on $M-K$, when $T > T_1$,

$$\int_M(\nabla\phi,\nabla\phi)\mathrm{dvol}\leq(\frac{b\epsilon_f}{2})^2T^2(1+\frac{4C_R}{(b\epsilon_f)^2T^2}+\frac{4C_K}{(b\epsilon_f)^2T})\int_K(\phi,\phi)\mathrm{dvol}. \tag{3.5}$$

Now define $Q : L \to L^2\Lambda^*(K)$, by $Qu = u|_K$. By (3.5), it's easy to see that $Q$ is injective, and $Im(Q) \subset W^{1,2}(\Lambda^*K)$. Since $W^{1,2}(\Lambda^*K) \hookrightarrow L^2\Lambda^*(K)$ is compact, $dim(L) = dim(Im(Q))$ must be finite. □

We now state the important consequence of this section. By combining Theorem 3.1.1 and Theorem 3.1.2 with Proposition A.2.1, decomposition (A.4), we have

**Theorem 3.1.3.** *Assume that $(M, g, f)$ is well tame. Then when $T > T_1$, we have the Kodaira decomposition*

$$L^2\Lambda^*(M) = \ker\Box_{Tf}\oplus\mathrm{Im}(\bar{d}_{Tf})\oplus\mathrm{Im}(\bar{\delta}_{Tf}).$$

*Furthermore, the Hodge Theorem holds:*

$$H^*_{(2)}(M, d_{Tf}) \cong \ker \Box_{Tf}.$$

**Remark 3.1.3.** *If $(M, g, f)$ is strongly tame, $T_1$ could be arbitrarily small, hence Theorem 3.1.3 holds true for any $T > 0$.*

## 3.2 The Agmon Estimate

In this section, we assume that $(M, g, f)$ is well tame, and $T > T_1$, where $T_1$ is described in Lemma 3.1.1.

Let $\tilde{g}_T := b^2T^2|\nabla f|^2 g$ be the Agmon metric on $M$. Let $K$ be the compact set as in last section. In this and later sections we define the Agmon distance $\rho_T(p)$ be the distance between $p$ and $K$ induced by $\tilde{g}_T$. Then we have $|\nabla \rho_T|^2 = b^2T^2|\nabla f|^2$ a.e. $p \notin K$, where the gradient $\nabla$ is induced by $g$.

For simplicity, denote $b^2T^2|\nabla f|^2$ by $\lambda_T$. We need the following two technical lemmas, whose proofs are postponed to Section 3.5.

**Lemma 3.2.1.** *Assume $w \in L^2(M), 0 \leq u \in L^2(M)$, and $(\Delta + \lambda_T)u \leq w$ outside the compact subset $K \subset M$ in the weak sense. That is*

$$\int_{M-K} \nabla u \nabla v + \lambda_T uv \mathrm{dvol} \leq \int_{M-K} w \cdot v \mathrm{dvol}, \quad \forall\ 0 \leq v \in C_c^\infty(M-K).$$

*Then for any $j \in \mathbb{N}$, there exists another compact subset $L \supset K$ of $M$ such that*

$$\begin{aligned}\int_{M-L} |u|^2 \lambda_T \exp(2b\rho_{T,j}) \mathrm{dvol} \leq C_2 &\int_{M-K} |w|^2 \lambda_T^{-1} \exp(2b\rho_{T,j}) \mathrm{dvol} \\ &+ C_1 \int_{L-K} |u|^2 \lambda_T \exp(2b\rho_{T,j}) \mathrm{dvol}\end{aligned} \tag{3.6}$$

*for* $C_1 = \frac{8(1+b^2)}{(1-b^2)^2}$, $C_2 = \frac{4}{(1-b^2)^2}$.

*Here* $\rho_{T,j} := \min\{\rho_T, j\}$.

**Corollary 3.2.1.** *If* $w = cu$ *for some* $c > 0$ *and* $T > \frac{2\sqrt{1+c}}{b\epsilon_f}$, *then*

$$I(u) := \int_M |u|^2 \exp(2b\rho_T)\mathrm{dvol} < \infty.$$

*Proof.* With this choice of $T$, $\lambda_T > 1 + c$ outside $K$. Now replacing $\lambda_T$ with $\lambda_T - c$ and $w$ with 0 in Lemma 3.2.1, we get

$$\int_{M-L} |u|^2 \exp(2b\rho_{T,j})\mathrm{dvol} \leq \int_{M-L} |u|^2(\lambda_T - c) \exp(2b\rho_{T,j})\mathrm{dvol}$$
$$\leq C_1 \int_{L-K} |u|^2\lambda_T \exp(2b\rho_{T,j})\mathrm{dvol} \leq C_1 \int_{L-K} |u|^2\lambda_T \exp(4b)\mathrm{dvol} < \infty.$$

Now let $j \to \infty$. By the monotone convergence theorem, we finish the proof. $\square$

By refining the argument above, we have the following corollary which will be used in the proof of our Agmon estimate for eigenforms.

**Corollary 3.2.2.** *If* $0 \leq u \in \mathrm{Dom}(\Box_{Tf})$, *and* $\Box_{Tf}u \leq (c + T|\nabla^2 f|)u$ *for some* $c > 0$ *and* $T > \max\{\sqrt{\frac{3}{b^2\epsilon_f}}, \sqrt{\frac{2C_2 c}{b^2\epsilon_f}}, 2c_f C_2\}$, *then*

$$I(u) := \int_M |u|^2 \exp(2b\rho_T)\mathrm{dvol} \leq CT^2\|u\|^2$$

*where the constant* $C = C(C^L, c_f, \epsilon_f, b, c), L = \{p \in M : \rho_T(p) \leq 2\}, C^L > \max_L |\nabla f|^2$.

*Proof.* Following the proof of Lemma 3.2.1 given in Section 7.1, put $L = \{p \in M : \rho_T(p) \leq 2\}$. Since $u \in \mathrm{Dom}(\Box_{Tf})$, $|\nabla f|u \in L^2(M)$.

Then by Lemma 3.2.1 we deduce

$$\int_{M-L} |u|^2 \lambda_T \exp(2b\rho_{T,j})\mathrm{dvol} \leq C_1 \int_{L-K} |u|^2 \lambda_T \exp(2b\rho_{T,j})\mathrm{dvol} + C_2 \int_{M-K} (c + T|\nabla^2 f|)\lambda_T^{-1}|u|^2 \exp(2b\rho_{T,j})\mathrm{dvol}$$

for $C_1, C_2$ as above.

Since $u \in L^2(M)$, $\int_{M-K}(c+T|\nabla^2 f|)\lambda_T^{-1}|u|^2 \exp(2b\rho_{T,j})\mathrm{dvol} < \infty$.

We split the second integral on the right hand side into two; the one over $L-K$ will be absorbed into the first term. The second term is (we omit the volume form here)

$$C_2 \int_{M-L} (c + T|\nabla^2 f|)\lambda_T^{-1}|u|^2 \exp(2b\rho_{T,j}) \leq (\frac{4C_2 c}{b^2 T^2 \epsilon_f^2} + \frac{2C_2 c_f}{T}) \int_{M-L} |u|^2 \exp(2b\rho_{T,j})$$

Combining the above one arrives at

$$\int_{M-L} |u|^2 \lambda_T \exp(2b\rho_{T,j}) \leq C_1 \int_{L-K} |u|^2 (\lambda_T + \frac{4C_2 c}{b^2 T^2 \epsilon_f^2} + \frac{2C_2 c_f}{T}) \exp(2b\rho_{T,j}) + C_2(\frac{4c}{b^2 T^2 \epsilon_f^2} + \frac{2c_f}{T}) \int_{M-L} |u|^2 \exp(2b\rho_{T,j}).$$

Thus,

$$\int_{M-L} |u|^2 (\lambda_T - \frac{4C_2 c}{b^2 T^2 \epsilon_f^2} + \frac{2C_2 c_f}{T}) \exp(2b\rho_{T,j}) \leq 2C_1 (C^L b^2 T^2 + \frac{4C_2 c}{b^2 T^2 \epsilon_f^2} + \frac{2C_2 c_f}{T}) e^{4b} \|u\|^2,$$

where $C^L > \max_L |\nabla f|^2$. If $T > \max\{2\frac{\sqrt{3}}{b\epsilon_f}, \frac{2\sqrt{C_2 c}}{b\epsilon_f}, 2c_f C_2\}$, then

$$\lambda_T - (\frac{4C_2 c}{b^2 T^2 \epsilon_f^2} + \frac{2C_2 c_f}{T}) > 1$$

outside $L$. Hence

$$\int_{M-L} |u|^2 \exp(2b\rho_{T,j})\mathrm{dvol} \leq 2C_1(C^L b^2 T^2 + C_2(\frac{2c}{b^2T^2\epsilon_f} + \frac{2c_f}{T}))e^{4b}\|u\|^2,$$

and consequently

$$\int_{M} |u|^2 \exp(2b\rho_{T,j})\mathrm{dvol} \leq [2C_1(C^L b^2 T^2 + \frac{2C_2 c}{b^2T^2\epsilon_f} + \frac{2C_2 c_f}{T}) + 1]e^{4b}\|u\|^2,$$

for $T > \max\{\frac{2\sqrt{3}}{b\epsilon_f}, \frac{2\sqrt{C_2 c}}{b\epsilon_f}, 2c_f C_2\}$.

Now let $j \to \infty$. By the monotone convergence theorem again, we finish the proof. $\square$

**Remark 3.2.2.** *It may seem that $C^L$ and $C_L$ depend on $T$ as $L = \{p \in M : \rho_T(p) \leq 2\}$. However, notice that as $T$ becomes bigger, $L$ gets smaller. Hence we can choose $C^L > \max_{p\in L} |\nabla f|(p)$, $C_L > \max_{p\in L} |\nabla^2 f(p)|$, which are then independent of $T$.*

**Lemma 3.2.3** (De Giorgi-Nash-Moser Estimates). *For $r > 0$, let $B_r(p)$ be the geodesic ball around $p$ with radius $r$ (in the metric $g$). Let $0 \leq u \in L^2(M)$, and $\Delta u \leq cu$ on $B_{2r}(p)$ in the weak sense for some constant $c \geq 0$. Then there exists constant $C_3(n, c, r_0, R) > 0$ depending only on the dimension $n$, the Sobolev constant (which depends on the injectivity radius lower bound $r_0$ and curvature bound on $R$) and $c$, such that for $r \leq r_0$*

$$\sup_{y\in B_r(p)} u(y) \leq \frac{C_3}{r^{n/2}}\|u\|_{L^2(B_{2r}(p))}.$$

With these preparation we are now ready to prove our first main estimate for the eigenforms of $\Box_{Tf}$.

*Proof of Theorem 1.3.1.* Consider an eigenform $\omega$ of $\Box_{Tf}$. That is $\Box_{Tf}\omega = \mu(T)\omega$, where the eigenvalue $\mu(T)$ satisfies $|\mu(T)| \leq c$ for some constant $c$. Then letting $u = g(\omega, \omega)^{1/2}$, by a straightforward computation using the Bochner's formula (for forms) and the Kato's inequality, we have

$$\Box_{Tf} u \leq (c + |R| + T|\nabla^2 f|)u,$$

where $|R|$ is the upper bound of curvature tensor. Hence by Corollary 3.2.2, we have, for $T \geq \max\{\frac{2\sqrt{3}}{b\epsilon_f}, \frac{2\sqrt{C_2(c+|R|)}}{b\epsilon_f}, 2c_f C_2\}$,

$$I(u) = \int_M |u|^2 \exp(2b\rho_T)\mathrm{dvol} \leq CT^2\|u\|^2$$

where the constant $C = C(C^L, c_f, \epsilon_f, b, c, |R|, n)$.

Recall that for the compact set $K$, (3.1) is satisfied. Hence by Proposition 1.2.5, the conditions of Lemma 3.2.3 are satisfied for $u$ on $M - K$. Namely, for $T > T_1$,

$$\Delta u \leq (c + |R|)u$$

on $M - K$. Also, the Agmon distance $\rho_T(p)$ is the distance between $p$ and $K$ induced by $\tilde{g}_T$ and $L = \{p \in M : \rho_T(p) \leq 2\}$. Suppose $p \in M - L$. Denote by $\tilde{B}_r(p)$ the $\tilde{g}_T$-geodesic ball around $p$ with radius $r$. Set $l = \sup_{q\in\tilde{B}_2(p)} |T\nabla f|(q)$, and $r = 1/(2l)$. Then one can easily verify that $B_{2r}(q) \subset \tilde{B}_2(p)$, $\forall q \in \tilde{B}_1(p)$.

Choose $q_0 \in \overline{\tilde{B}_2(p)}$ so that $|T\nabla f|(q_0) \in (l/2, l]$. By Lemma 3.2.1 and de Giorgi-Nash-Moser estimate (Lemma 3.2.3), we have

$$\begin{aligned}
|u(p)|^2 \exp(2b\rho_T(p)) &\leq \frac{C_3(n, c, r_0, R)}{r^n}\|u\|^2_{L^2(B_{2r}(p))} \exp(2b\rho_T(p)) \\
&\leq \frac{C_4(n, c, r_0, R)}{r^n} \int_{\tilde{B}_2(p)} |u|^2(q) \exp(2b\rho_T(q))\mathrm{dvol} \\
&\leq C_5(C^L, c_f, \epsilon_f, b, c, R, n, r_0)|T\nabla f(q_0)|^n I(u).
\end{aligned}$$

We will prove that

$$|\nabla f(q_0)|^2 \leq \sup_{p' \in K} |\nabla f|^2(p') \exp(\frac{2c_f}{bT}\rho_T(q_0)) \tag{3.7}$$

in Lemma 3.3.6. Hence,

$$|\nabla f(q_0)|^2 \leq \sup_{p' \in K} |\nabla f|^2(p') \exp(\frac{2c_f}{bT}\rho_T(q_0)) \leq \sup_{p' \in K} |\nabla f|^2(p') \exp(\epsilon\rho_T(q_0))$$

for any small $\epsilon$, provided $T \geq \frac{2c_f}{b\epsilon}$. It follows then that,

$$|u(p)|^2 \leq C_6(C^L, c_f, \epsilon_f, a, b, c, R, n, r_0) I(u) T^n \exp(-2a\rho_T(p)),$$

for any $a < b$ provided $T \geq \frac{nc_f}{b(b-a)}$. Hence if

$$T \geq T_2(K) := \max\{\frac{2\sqrt{3}}{b\epsilon_f}, \frac{2\sqrt{C_2(c+|R|)}}{b\epsilon_f}, 2c_f C_2, \frac{nc_f}{b(b-a)}\}, \tag{3.8}$$

$$|u(p)|^2 \leq C_7(C^L, c_f, \epsilon_f, a, b, c, r_0, |R|, n) T^{n+2} \exp(-2a\rho_T(p)) \|u\|^2.$$

$\square$

**Remark 3.2.4.** *The proof above gives the inequality for $p \in M - L = \{\rho_T(p) > \tau_0\}$ for some constant $\tau_0$ independent of $T$, which is what we needed for later applications. For $p \in L$, using the same reasoning as in Remark 3.2.2, there exist constant $C > 0$, which is independent of $T$, such that*

$$\Delta u \leq CTu. \tag{3.9}$$

*for all $p \in L$. Therefore via Moser iteration as in Lemma 3.2.3 and similar arguments as above, one can show that*

$$|u|^2(p) \leq C'T^n \|u\|^2_{L^2} \leq C' \exp(2a) T^n \exp(-a\rho_T) \|u\|^2_{L^2}.$$

**Remark 3.2.5.** *When $(M, g, f)$ is strongly tame, $T_2$ can be arbitrarily small.*

## 3.3 Thom-Smale Theory

In this and the next section, we assume that $f$ is a Morse function on $M$. Moreover, let $K$ be a suitable compact subset of $M$ such that $\epsilon_f(K) > 0$, $\epsilon > 0$ be small enough (to be determined later), $T_5(\epsilon, K) := \frac{c_f}{\epsilon}$. Then outside $K$, we have

$$T|\nabla^2 f| \leq \epsilon T^2 |\nabla f|^2, \tag{3.10}$$

provided $T \geq T_5$.

**Remark 3.3.1.** *We can take $T_5$ to be arbitrarily small if $(M, g, f)$ is strongly tame.*

In this section, we always assume that $T \geq T_5$. Under these conditions we will define the Thom-Smale complex $(C_*(W^u), \tilde{\partial})$ but leave the proof that $\tilde{\partial}^2 = 0$ to Section 3.5.3. The remaining of the Section is devoted to the pairing between the Thom-Smale complex and the Witten instanton complex.

Before defining the Thom-Smale complex, there is still a subtle issue for noncompact cases. That is, the gradient vector field $-\nabla f$ may not be complete, i.e., its flow curves may not exist for all time. But notice that if we rescale the vector field by some positive function, the corresponding integral curves will simply be reparameterizations of the original integral curves.

For this purpose, we fix a positive smooth function $F$ such that,

$$F|_{M-K} = \frac{1}{bT|\nabla f|^2}$$

Then we have

**Lemma 3.3.2.**

*(1)*

$$bT|f(p)-f(q)| \leq \tilde{d}_T(p,q), \quad \forall p,q \in M. \tag{3.11}$$

*(2) Let $\tilde{\Phi}^t$ be the flow generated by $Y_f := -F\nabla f$, and $p \in M$. If the flow line $\tilde{\Phi}^t(p), t \in [s_1, s_2]$ is outside $K$, then it's the minimal geodesic connecting $\tilde{\Phi}^{s_1}(p)$ and $\tilde{\Phi}^{s_2}(p)$ with respect to metric $\tilde{g}_T$. Moreover,*

$$\tilde{d}_T(\tilde{\Phi}^{s_1}(p), \tilde{\Phi}^{s_2}(p)) = bT|f(\tilde{\Phi}^{s_1}(p)) - f(\tilde{\Phi}^{s_2}(p))| = |s_2 - s_1|. \tag{3.12}$$

*(3) $Y_f$ is a complete vector field.*

*Proof.* (1). Let $\gamma : [0, \tilde{d}_T(p,q)] \to M$ be the minimal geodesic connecting $p$ and $q$ with respect to $\tilde{g}_T$. Let $\tilde{\nabla}^T$ be the Levi-Civita connection induced by $\tilde{g}_T$. One computes

$$\begin{aligned}
\frac{d}{ds} f(\gamma(s)) &= \tilde{g}_T(\tilde{\nabla}^T f, \gamma'(s)) = \tilde{g}_T(\frac{\nabla f}{b^2T^2|\nabla f|^2}, \gamma'(s)) \\
&\leq \sqrt{\tilde{g}_T(\frac{\nabla f}{b^2T^2|\nabla f|^2}, \frac{\nabla f}{b^2T^2|\nabla f|^2})} = \frac{1}{bT}.
\end{aligned}$$

Consequently, $bT|f(p)-f(q)| \leq \tilde{d}_T(p,q)$.

(2). We give a direct proof (See [38] Lemma A 2.2 for another).

Let's first show that $\gamma(s) := \tilde{\Phi}^s(p), s \in [s_1, s_2]$ is a geodesic. Since $\tilde{g}_T(Y_f, Y_f) = 1$ outside $K$, we let $\tilde{e}_1^T(s), ..., \tilde{e}_n^T(s)$ be a local orthomormal frame on $\gamma$ with $\tilde{e}_1^T = Y_f$. One can easily show that outside $K$, $-Y_f/(bT)$ is the gradient of $f$ with respect to $\tilde{g}_T$. In order to prove $\gamma'' = 0$, it suffices to prove $\tilde{g}_T(\gamma'', \tilde{e}_i^T) = 0, i \geq 2$.

Indeed,

$$\begin{aligned}
\tilde{g}_T(\gamma'', \tilde{e}_i^T) &= \tilde{g}_T(\tilde{\nabla}^T_{Y_f} Y_f, \tilde{e}_i^T) \\
&= -\tilde{g}_T(Y_f, [Y_f, \tilde{e}_i^T]) \\
&= [Y_f, \tilde{e}_i^T] f \\
&= -Y_f \tilde{g}_T(\tilde{e}_i^T, \frac{Y_f}{bT}) + \tilde{e}_i^T \tilde{g}_T(Y_f, \frac{Y_f}{bT}) \\
&= 0.
\end{aligned}$$

We now prove that $\gamma$ is the shortest geodesic connecting $\gamma(s_1)$ and $\gamma(s_2)$ in $(M, \tilde{g}_T)$. Assume that $\sigma : [s_1, s_2'] \mapsto M$ is another normal geodesic connecting $\gamma(s_1)$ and $\gamma(s_2)$ induced by $\tilde{g}_T$. Then $\tilde{g}_T(\sigma'(s_1), Y_f) < 1$. Set $\alpha(s) = f \circ \gamma(s), \beta(s) = f \circ \sigma(s)$, then we have $\alpha(s_1) = (s_1)$, and $\alpha'(s) = -1, '(s) = -\tilde{g}_T(\sigma'(s), Y_f \circ \gamma(s)) \geq -1$. Hence by a comparison theorem in ODE, we must have $\alpha(s) \leq (s)$. Now $\sigma(s_2') = \gamma(s_2)$, and $s_2' - s_1 = Length(\sigma)$. Thus, $a(s_2') \leq (s_2') = \alpha(s_2)$. Since $\alpha$ is decreasing, we must have $s_2' \geq s_2$.

Therefore, $\tilde{\Phi}^s(y), s \in [s_1, s_2]$ is one of shortest geodesic connecting $y$ and $\tilde{\Phi}^t(y)$.

Hence $\tilde{d}_T(\tilde{\Phi}^{s_1}(p), \tilde{\Phi}^{s_2}(p)) = |s_2 - s_1|$. Moreover, since $\frac{\partial}{\partial s} f(\tilde{\Phi}^s(p)) = Y_f f = g(\nabla f, Y_f) = \frac{1}{bT}$,

$$bT|f(\tilde{\Phi}^{s_1}(p)) - f(\tilde{\Phi}^{s_2}(p))| = |s_2 - s_1|.$$

(3). To prove that $Y_f$ is complete, we show that for any $p \in M$, there exists a uniform constant $\epsilon_0 > 0$ such that $\tilde{\Phi}^t(p)$ is well defined on $(-\epsilon_0, \epsilon_0)$.

Recall that $L := \{p \in M : \tilde{d}_T(p, K) \leq 2\}$. It suffices to show that for any $p \in M - L$, $\tilde{\Phi}^t(p)$ is well defined on $(-1, 1)$, as $L$ is compact.

But this is clear: on $M - K$, $bTF^{-1} g(Y_f, Y_f) = \tilde{g}_T(Y_f, Y_f) = 1$, and $(M, bTF^{-1}g)$ is complete, and $\tilde{\Phi}^t(p)$, $t \in (-1, 1)$ is a geodesic inside $M - K$ with respect to $bTF^{-1}g$.

□

Now we can talk about the unstable and stable manifolds of $Y_f$:

Let $x$ be a critical point of the Morse function $f$, $W^s(x)$ and $W^u(x)$ be the stable and unstable manifold of $x$ with respect to flow $\tilde{\Phi}^t$ defined in Lemma 3.3.2 (See Chapter 6 in [27] for a precise definition of stable and unstable manifolds). We will further assume that $f$ satisfies the Smale transversality condition, namely $W^s(x)$ and $W^u(y)$ intersect transversally. Then the Thom-Smale complex $(C_*(W^u), \tilde{\partial})$ is defined by

$$C_*(W^u) = \oplus_{x\in \mathrm{Crit}(f)} \mathbb{R} W^u(x),$$

and

$$C_i(W^u) = \oplus_{x\in \mathrm{Crit}(f), n_f(x)=i} \mathbb{R} W^u(x).$$

To define the boundary operator, let $x$ and $y$ be critical points of $f$, with $n_f(y) = n_f(x) - 1$.

For $x \in \mathrm{Crit}(f)$, set

$$\tilde{\partial} W^u(x) = \sum_{y\in \mathrm{Crit}(f), n_f(y)=n_f(x)-1} m(x,y) W^u(y).$$

Here the integer $m(x,y)$ is the signed counts of the flow lines in $W^s(y) \cap W^u(x)$.

In order to see that the integer $m(x,y)$, and hence the coboundary operator, is well defined, we now make a more judicious choice of $K$. Fix any $p_0 \in M$. Let $\tilde{d}$ be the distance function induced by (the Agmon metric) $|\nabla f|^2 g$, and set

$$D = \sup_{y\in \mathrm{Crit}(f)} \tilde{d}(y, p_0) + 2 \sup_{y,z\in \mathrm{Crit}(f)} \tilde{d}(y,z). \tag{3.13}$$

We choose $K$ so that

$$\tilde{B}_{D+1}(p_0) \subset K^\circ,$$

where $K^\circ$ denotes the interior of $K$, $\tilde{B}_r(p_0) := \{p \in M : \tilde{d}(p, p_0) \leq r\}$. From the definition of $D$, it's clear that all critical points are contained in $K^\circ$. Moreover we make the following remark.

**Remark 3.3.3.** *The choice of $K$ together with (3.12) guarantees that for any $x, y \in \mathrm{Crit}(f)$, $W^s(x) \cap W^u(y) \subset K^\circ$. See also Lemma 3.5.6 for more detail.*

*Thus, just like the compact case, by the transversality, $m(x, y)$ is well defined.*

We will prove in Section 3.5.3 that under our tameness condition, $\tilde{\partial}^2 = 0$. Thus, $(C_*(W^u), \tilde{\partial})$ is a complex.

Let $F_{Tf}^{[0,1],*}$ be the space spanned by the eigenforms of $\Box_{Tf}$ with eigenvalue lying in $[0, 1]$. By Theorem 3.1.2, $F_{Tf}^{[0,1],*}$ is finite dimensional when $T$ is big enough. By previous discussions, the cohomology of the Witten instanton complex is $H^*_{(2)}(M, d_{Tf})$ when $T$ is large enough.

To prove Theorem 1.3.3, we now consider the following chain map $\mathcal{J} : (F_{Tf}^{[0,1],*}, d_{Tf}) \mapsto C^*((W^u)', \tilde{\partial}')$. Here $C^*((W^u)', \tilde{\partial}')$ denote the dual chain complex. Let $W^u(x)'$ be the dual basis of $W^u(x)$. Then

$$\mathcal{J}\omega = \sum_{x \in \mathrm{Crit}(f)} W^u(x)' \int_{W^u(x)} \exp(Tf)\omega.$$

However there is a technical issue here we need to address. When $\overline{W^u(x)}$ is compact, the integral $\int_{W^u(x)} \exp(Tf)\omega$ is clearly well defined, but $\overline{W^u(x)}$ here may be noncompact. We will be content here only with the well-definedness of the map and leave the proof that $\mathcal{J}$ is indeed a chain map to Section 7.3, see Corollary 3.5.2.

Let $r > 0$ small enough, $B_r^{n_f(x)}(x) \subset K$ be the $n_f(x)$-dimensional ball in $W^u(x)$ with center $x$ and radius $r$ with respect to metric $g$. As before, let $\tilde{\Phi}^t$ be the flow generated by $-F\nabla f$. Then $W^u(x) = \cup_{t>0} \tilde{\Phi}^t(B_r^{n_f(x)}(x))$. Moreover, by the definition of unstable manifold, if $t_1 < t_2$, $\tilde{\Phi}^{t_1}(B_r^{n_f(x)}(x)) \subset \tilde{\Phi}^{t_2}(B_r^{n_f(x)}(x))$.

Therefore, for any $\omega \in F_{Tf}^{[0,1],*}$

$$\begin{aligned}&\left|\int_{W^u(x)} \exp(Tf)\omega\right| = \left|\lim_{t\to\infty}\int_{(\tilde\Phi^t)(B_r^{n_f(x)}(x))} \exp(Tf)\omega\right|\\&\le C\exp(Tf(x))\lim_{t\to\infty}\int_{B_r^{n_f(x)}(x)} |\omega|\circ\tilde\Phi^t|\det((\tilde\Phi^t)_*)|\mathrm{dvol}_{W^u(x)}\end{aligned}$$

The well-definedness of $\mathcal{J}$ is now reduced to the following two technical lemmas, as well as Theorem 1.3.1 and the well tameness of $(M, g, f)$.

**Lemma 3.3.4.** *Suppose $t > 0$ is big enough, $y \in B_r^{n_f(x)}(x) - \tilde\Phi^{-t}K$. Then*

$$|\rho_T(\tilde\Phi^t(y)) - t| < T\sup_{p\in K}|\nabla f|\,\mathrm{diam}(K),$$

*where* $\mathrm{diam}(K)$ *is the diameter of $K$ with respect to metric $g$.*

*Proof.* For any $y \in B_r^{n_f(x)}(x) - \tilde\Phi^{-t}K$, (3.12) and the triangle inequality give

$$\begin{aligned}&|\rho_T(\tilde\Phi^t(y)) - t| = |\tilde d_T(\tilde\Phi^t(y), K) - \tilde d_T(\tilde\Phi^t(y), y)|\\&\le T\sup_{p\in K}|\nabla f|\mathrm{diam}(K),\end{aligned}$$

where $\tilde d_T$ is the distance induced by $\tilde g_T$. ☐

**Lemma 3.3.5.** *Fix any $y \in B_r^{n_f(x)}(x) - \tilde\Phi^{-t}K$ and set $p = \tilde\Phi^t(y)$. we have*

$$|\tilde\Phi^t_*(y)| \le C_7(T)\exp(\frac{6\epsilon\rho_T(p)}{b}).$$

*Hence,*

$$|\det(\Phi^t)_*(y)| \le C_7(T)\exp(\frac{6n_f(x)\epsilon\rho_T(p)}{b}).$$

*Here $C_7$ is a constant independent of $y$. In particular, for the fixed $a \in (0,1)$ in Theorem 1.3.1 and $b \in (a,1)$, any choice of $0 < \epsilon \leq \frac{ab}{12n}$ will guarantee that $\mathcal{J}$ is well defined for $T > T_5(\epsilon)$.*

*Proof.* Let $e$ be a unit tangent vector of $W^u(x)$ at $y$, extend $e$ to a local unit vector field (still denoted by $e$) of $W^u(x)$ near $y$. Noting that from (3.10)

$$|\nabla_{\tilde{\Phi}^t_* e}(\tilde{\Phi}^t)_*(Y_f)| \leq \frac{3\epsilon}{b}|(\tilde{\Phi}^t)_* e|,$$

we have

$$\begin{aligned}
&|\frac{\partial}{\partial t} g((\tilde{\Phi}^t)_* e(y), (\tilde{\Phi}^t)_* e(y))| \\
&= 2|g(\nabla_{(\tilde{\Phi}^t)_* e(y)}(\tilde{\Phi}^t)_* Y_f, (\tilde{\Phi}^t)_* e(y))| \\
&\leq \frac{6\epsilon}{b}|g((\tilde{\Phi}^t)_* e(y), (\tilde{\Phi}^t)_* e(y))|.
\end{aligned}$$

By a classical result in ODE, we have

$$g((\tilde{\Phi}^t)_* e(y), (\tilde{\Phi}^t)_* e(y)) \leq C_8 \exp(\frac{6\epsilon t}{b}).$$

Our lemma follows from Lemma 3.3.4. $\square$

Now when $(M, g, f)$ is well tame, we set $T_0$ to be the smallest nonnegative number, such that $\forall \delta > 0$,

1. whenever $T \geq T_0 + \delta$, Theorem 1.3.1 holds true for the Agmon distance with respect to some compact subset $K(\delta) \subset M$ depending on $\delta$;
2. Theorem 3.1.3 holds true whenever $T > T_0$;
3. The map $\mathcal{J}$ is well defined whenever $T > T_0$.

Fix a compact set $K$ as before. Then

$$T_0 \leq \max\{T_1(K), T_2(K), T_5(K)\}. \tag{3.14}$$

(Cf. (3.3) for the description of $T_1$, (3.8) for $T_2$.) Moreover, if $(M, g, f)$ is strongly tame, $T_0 = 0$.

We note in passing the following lemma which plays an important role in estimating the eigenforms previously.

**Lemma 3.3.6.** *Suppose $T \geq T_0$. Then for any $q \in M$,*

$$|\nabla f|^2(q) \leq \sup_{p \in K} |\nabla f|^2(p) \exp(\frac{2c_f}{bT}\rho_T(q))$$

*Proof.* Let $\gamma : [0, \rho_T(q)] \mapsto M$ be a normal minimal $\tilde{g}_T$-geodesic connecting $K$ and $q$. Then we have $g(\gamma', \gamma') = \frac{1}{b^2T^2|\nabla f|^2}$ outside $K$.

Let $h(t) = |\nabla f|^2 \circ \gamma$, then

$$h'(t) = 2g(\nabla_{\gamma'}\nabla f, \nabla f) \leq \frac{2}{bT}|\nabla^2 f| \leq \frac{2c_f}{bT}|\nabla f|^2 = \frac{2c_f}{bT}h(t),$$

Hence $|\nabla f|^2(q) \leq |\nabla f|^2 \circ \gamma(0) \exp(\frac{2c_f}{bT}\rho_T(q))$. $\square$

## 3.4 Morse Inequalities

In this section, we assume that $T \geq T_0$. In fact, we also assume that in a neighborhood $U_x$ of critical points $x$ of $f$, we have coordinate system $z = (z_1, ..., z_n)$, such that for $k = n_f(x)$,

$$f = f(x) - z_1^2 - ... - z_k^2 + z_{k+1}^2 + ... + z_n^2, \quad g = dz_1^2 + ... + dz_n^2. \tag{3.15}$$

This is a generic condition. Without loss of generality we assume that $U_x$ is an Euclidean open ball around $x$ with radius 1. Also, these open sets are disjoint.

Recall that $m_i$ denotes the number of critical points of $f$ with Morse index $i$. We have the following proposition.

**Proposition 3.4.1.** *There exists $T_3 \geq T_0$ big enough (see (3.14) for the definition of $T_0$)), so that whenever $T \geq T_3$, the number of eigenvalues (counted with multiplicity) in $[0,1]$ of $\Box_{Tf}|_{\Omega^i_{(2)}(M)}$ equals $m_i$. I.e.* $\dim F^{[0,1],*}_{Tf} = m_i$.

The proof of Proposition 3.4.1 follows from that of Proposition 5.5 in [27], except for the proof of the following proposition, which requires a slight modification using the well tame condition.

**Proposition 3.4.2.** *There exist constants $C > 0$, $T_4 > 0$ such that for any smooth form $\phi \in \Omega^*_{(2)}(M)$ with $\mathrm{supp}(\phi) \subset M - \cup_{x\in \mathrm{Crit} f} U_x$ and $T \geq T_4$, one has*

$$\|\Box_{Tf}\phi\|_{L^2} \geq CT\|\phi\|_{L^2}.$$

*Here* $\mathrm{supp}(\phi)$ *denotes the support of* $\phi$.

*Proof.* Since $f$ is well tame, there exist $\delta_1, \delta_2 > 0$, such that $|\nabla f| \geq \delta_1$ and $|\nabla^2 f| \leq \delta_2 |\nabla f|^2$ on $M - \cup_{x\in \mathrm{Crit} f} U_x$. Then our proposition follows from the same argument in Proposition 4.7 of [27]. $\Box$

On the other hand, $(F^{[0,1],*}_{Tf}, d_{Tf})$ form a complex, the so called Witten instanton complex, whose cohomology is $H^*_{(2)}(M, d_{Tf})$, when $T$ is big enough, by Theorem 3.1.3. As a result, our Theorem 1.3.2 (the strong Morse inequalities) follows from Proposition 3.4.1 and our Hodge theorem when $T > T_3$. For the case of $T \in (T_0, T_3]$, see Section 3.5.

## 3.5 The Agmon Estimate—Technical Parts

In this section we first carry out the main technical estimates of the paper. Then, in Section 3.5.3, we establish the Stokes formula for the Thom-Smale complex in our setting and deduce among its consequences that the square of the coboundary operator for the Thom-Smale complex is zero. The remaining subsections are devoted to the rest of the proof for Theorem 1.3.3 and Theorem 1.3.1.

### 3.5.1 Proof of Lemma 3.2.1

*Proof.* Our proof is adapted from that of Theorem 1.5 in [32].

Let $L = \{p \in M : \rho_T(p) \leq 2\}$. Let $\eta_k \in C_c^\infty(\mathbb{R})$ ($k$ large enough) be a smooth bump function such that

$$\eta_k(t) = \begin{cases} 0, \text{ If } |t| < 1 \text{ or } |t| > k+1; \\ 1, \text{ If } |t| \in (2,k), \end{cases}$$

and $|\eta_k'(t)| \leq 2$, $\eta_k(t) \in [0,1], \forall t \in \mathbb{R}$.

Set $\rho_{T,j} = \min\{\rho_T, j\}$, and

$$\lambda_{T,j} = \begin{cases} \lambda_T, \text{ if } \rho_T < j, \\ 0, \text{ otherwise} \end{cases}.$$

Clearly $|\nabla \rho_{T,j}|^2 = \lambda_{T,j}$ a.e. and $\lambda_T \geq \lambda_{T,j}$.

Now set $\varphi_{k,j} = (\eta_k \circ \rho_T)\exp(b\rho_{T,j})$. Then by assumption, we have

$$\int_M \nabla u \nabla(\varphi_{k,j}^2 u) + \lambda_T (u\varphi_{k,j})^2 \mathrm{dvol} \leq \int_M w\varphi_{k,j}^2 u \mathrm{dvol}.$$

Noting that $\nabla u \nabla(\varphi_{k,j}^2 u) = |\nabla(\varphi_{k,j} u)|^2 - |\nabla\varphi_{k,j}|^2 u^2 \geq -|\nabla\varphi_{k,j}|^2 u^2$, we have

$$\int_{M-K} (\lambda_T |u\varphi_{k,j}|^2 - |u|^2 |\nabla\varphi_{k,j}|^2)\mathrm{dvol} \leq \int_{M-K} wu\varphi_{k,j}^2 \mathrm{dvol}. \tag{3.16}$$

Since (we now omit the volume form dvol in what follows)

$$\int_{M-K} wu\varphi_{k,j}^2 \leq \frac{1}{1-b^2}\int_{M-K} (\lambda_T)^{-1} w^2 \varphi_{k,j}^2 + \frac{1-b^2}{4}\int_{M-K} \lambda_T u^2 \varphi_{k,j}^2,$$

and

$$\begin{aligned} |\nabla\varphi_{k,j}|^2 &\leq \frac{1+b^2}{2}(\eta_k \circ \rho_T)^2 |\nabla\rho_{T,j}|^2 \exp(2b\rho_{T,j}) + \frac{1+b^2}{1-b^2}(\eta_k' \circ \rho_T)^2 |\nabla\rho_T|^2 \exp(2b\rho_{T,j}) \\ &= \frac{1+b^2}{2}(\eta_k \circ \rho_T)^2 \lambda_{T,j} \exp(2b\rho_{T,j}) + \frac{1+b^2}{1-b^2}(\eta_k' \circ \rho_T)^2 \lambda_T \exp(2b\rho_{T,j}), \end{aligned}$$

by (3.16), we have

$$\begin{aligned} &\frac{3+b^2}{4}\int_{M-K} \lambda_T (\eta_k \circ \rho_T)^2 u^2 \exp(2b\rho_{T,j}) - \frac{1+b^2}{2}\int_{M-K} \lambda_{T,j}(\eta_k \circ \rho_T)^2 u^2 \exp(2b\rho_{T,j}) \\ &\leq \frac{1}{1-b^2}\int_{M-K} w^2 (\eta_k \circ \rho_T)^2 \lambda_T^{-1} \exp(2b\rho_{T,j}) \\ &+ \frac{1+b^2}{1-b^2}\int_{M-K} u^2 (\eta_k' \circ \rho_T)^2 \lambda_T \exp(2b\rho_{T,j}) \\ &\leq \frac{1}{1-b^2}\int_{M-K} w^2 \lambda_T^{-1} \exp(2b\rho_{T,j}) \\ &+ 2\frac{1+b^2}{1-b^2}\int_{L-K} u^2 \lambda_T \exp(2b\rho_{T,j}) + 2\frac{1+b^2}{1-b^2}\int_{\tilde{B}_{k+1}-\tilde{B}_k} u^2 \lambda_T \exp(2bj) \end{aligned} \tag{3.17}$$

Letting $k \to \infty$, by the monotone convergence theorem and the fact that $\int_M \lambda_T u^2 < \infty$, we have

$$\frac{3+b^2}{4}\int_{M-L}\lambda_T u^2\exp(2b\rho_{T,j})-\frac{1+b^2}{2}\int_{M-L}\lambda_{T,j}u^2\exp(2b\rho_{T,j})$$
$$\leq\frac{1}{1-b^2}\int_{M-K}w^2\lambda_T^{-1}\exp(2b\rho_{T,j})+2\frac{1+b^2}{1-b^2}\int_{L-K}u^2\lambda_T\exp(2b\rho_{T,j}).$$

$\square$

### 3.5.2 Proof of Lemma 3.2.3

*Proof.* The proof is a standard argument of Moser iteration and we present it here for reader's convenience.

The starting point is the differential inequality

$$cu\geq\Delta u \tag{3.18}$$

weakly on $B_{2r}(p)$.

Set $r_1=2r, r_{k+1}=r_k-(1/2)^k r, n_k=(n/(n-2))^{k-1}$

Let $\eta_k\in C_c^\infty(B_{2r})$ be bump functions s.t.

$$\eta_k=\begin{cases}1 \text{ on } B_{r_{k+1}},\\ 0 \text{ on } B_{2r}-B_{r_k},\end{cases}$$

and $|\nabla\eta_k(q)|<\frac{2}{r_{k+1}-r_k}$, $\eta_k(q)\in[0,1],\forall q\in B_{2r}$.

Set $u_m=\min\{u,m\}$, and $\phi_1=\eta_1^2u_m\in H_0^1(B_{2r})$. Notice that $\phi_1=0$ and $\nabla\phi_1=0$ in

$\{u \geq m\}$. Hence, by (3.18), we have

$$\begin{aligned}
&\int_{B_{r_1}} c(u_m)^2 dvol \geq \int_{B_{2r}} cu\phi_1 dvol \geq \int_{B_{2r}} \nabla u \nabla \phi_1 dvol \\
&= \int_{B_{2r}} \eta_1^2 |\nabla u_m|^2 + 2\eta_1 \nabla \eta \nabla u_m u_m dvol \\
&\geq \int_{B_{2r}} \eta_1^2 |\nabla u_m|^2 - 1/2\eta_1^2 |\nabla u_m|^2 - 2|\nabla \eta u_m|^2 dvol \\
&\geq \int_{B_{2r}} \eta_1^2 |\nabla u_m|^2 - 1/2\eta_1^2 |\nabla u_m|^2 - 2|\nabla \eta u_m|^2 dvol \\
&\geq 1/2 \int_{B_{r_2}} |\nabla u_m|^2 dvol - 4/(r_2 - r_1)^2 \int_{B_{r_1}} |u_m|^2 dvol
\end{aligned}$$

Hence, we have

$$\int_{B_{r_2}} |\nabla u_m|^2 dvol \leq (2c + 8/(r_1 - r_2)^2) \int_{B_{r_2}} c(u_m)^2 dvol \leq C(n)/(r_1 - r_2)^2 \int_{B_{r_2}} c(u_m)^2 dvol.$$

By Sobolev inequality,

$$(\int_{B_{r_1}} |u_m|^{2n_2} dvol)^{1/n_2} \leq C(n)/(r_1 - r_2)^2 \int_{B_{r_2}} c(u_m)^2 dvol.$$

That is

$$\|u_m\|_{L^{2n_2}(B_{r_2})} \leq (C(n)/(r_1 - r_2)) \|u_m\|_{L^{2n_1}(B_{r_1})}$$

Let $m \to \infty$, we have

$$\|u\|_{L^{2n_2}(B_{r_2})} \leq (C(n)/(r_1 - r_2)) \|u\|_{L^{2n_1}(B_{r_1})}$$

Consider $\phi_k = \eta_k^2(u_m^{2n_k-1}) \in H_0^1(B_{2r})$. By the same arguments as above, we have

$$\|u\|_{L^{2n_{k+1}}(B_{r_{k+1}})} \leq (C(n)/(r_k - r_{k+1}))^{1/(n_k)}\|u\|_{L^{2n_k}(B_{r_k})}.$$

As a consequence,

$$\begin{aligned}
&\|u\|_{L^\infty(B_r)} = \lim_{k\to\infty}\|u\|_{L^{2n_k}(B_{r_k})} \\
&\leq C\Pi_{k=1}^{\infty}(C(n)/(r_k - r_{k+1}))^{1/(n_k)}\|u\|_{L^2(B_{2r})} \\
&= C(C(n)/r)^{(\sum_{k=1}^{\infty}1/(n_k))}2^{\sum_{k=1}^{\infty}k/n_k}\|u\|_{L^2(B_{2r})} \\
&\leq C/r^{n/2}\|u\|_{L^2(B_{2r})}.
\end{aligned}$$

□

We state two Lemmas that will be needed shortly,

**Lemma 3.5.1.** *Suppose that $u \in L^2(M)$, $w \in L^N(M)$ for some $N > n/2$, and $\Box_{Tf}u \leq w$ in the weak sense (and $u \geq 0$.). For $r > 0$ small enough, $p \notin L$, let $B_r(p)$ be the geodesic ball around $p$ with radius $r$ induced by $g$. Then*

$$\sup_{y\in B_r(p)} u(y) \leq C_2(r^{-n/2}\|u\|_{L^2(B_{2r}(p))} + r^{-n/N}\|w\|_{L^N(B_{2r}(p))}),$$

*where $C_2 > 0$ is a constant that depends only on the dimension $n$, the injectivity radius lower bound $r_0$ and the curvature bound.*

*Proof.* The proof is actually similar to the proof Lemma 3.2.3, requiring only some slight modification. See Theorem 4.1 in [41] for a reference. □

By the same argument as the proof of Theorem 1.3.1, we have

**Lemma 3.5.2.** *Let $(M, g, f)$ be well tame, $w \in L^N(M) \cap L^{N^*}(M)$ satisfying*

$$\|w\|^N_{L^N_{wt}} := \int_M |w|^N \exp(2a'' N \rho_T) dvol < \infty$$

*for some $a'' \in (0, b)$, where $N^* = \frac{N}{N-1}$. If $\phi \in L^2(M)$ is a weak solution of $\Box_{Tf}\phi \leq w$, then*

$$|\phi(p)| \leq C \left( \|\phi\|_{L^2} + \sup\{\|w\|_{L^{N^*}}, 1\} \|w\|_{L^N_{wt}} \right) \exp(-a'' \rho_T(p)).$$

### 3.5.3 On the Thom-Smale Complex

In this subsection, we will show that the Thom-Smale complex defined previously in Section 3.3 is indeed a complex. The key here is to establish the analog of the so called Stokes formula in our setting. We use a doubling construction to reduce it to the compact case and makes essential use of the uniform lower bound of $|\nabla f|$ outside suitably chosen compact sets, which guarantees that the flow lines coming out of the compact region will never return; see also Remark 7.7.

Intuitively the idea may be explained as follows. When the Morse function $f$ is proper, such compact regions can be chosen to be the sublevel set $a \leq f \leq b$. Since $f$ decreases along its negative gradient flow, flow line out of the region will obviously not return. In general, however, $f$ may not be proper, but it turns out that the Agmon distance is a good replacement. Indeed, when $f$ is proper, $f$ measures the Agmon distance between its level sets.

First, let's recall the Stokes formula in the compact case. The following is a restatement of Proposition 6 in [42].

**Proposition 3.5.3.** *Let $(N, g)$ be a compact Riemannian manifold (without boundary), and $f$ a Morse function. Assume that $(N, g, f)$ satisfies Thom-Smale transversality condition. Then, for any critical point $x \in \mathrm{Crit}(f)$ with Morse index $n_f(x)$, any*

$\phi \in \Omega^{n_f(x)-1}(M)$, *one has the following so called Stokes Formula*

$$\int_{W^u(x)} d\phi = \sum_{y\in \mathrm{Crit}(f), n_f(y)=n_f(x)-1} m(x,y)\int_{W^u(y)} \phi.$$

For our noncompact case with tame conditions and Thom-Smale transversality, we have similarly

**Proposition 3.5.4.** *For any critical point $x \in \mathrm{Crit}(f)$ with Morse index $n_f(x)$, any $\phi \in \Omega_c^{n_f(x)-1}(M)$, one has the following so called Stokes Formula*

$$\int_{W^u(x)} d\phi = \sum_{y\in \mathrm{Crit}(f), n_f(y)=n_f(x)-1} m(x,y)\int_{W^u(y)} \phi.$$

Before giving the proof of this proposition, we first draw a couple of consequences.

**Corollary 3.5.1.** *Let $\tilde{\partial} : C_*(W^u) \mapsto C_{*-1}(W^u)$ be the map constructed in Section 3.3, then $\tilde{\partial}^2 = 0$.*

*Proof.* Otherwise, $\tilde{\partial}^2 W^u(x) \neq 0$. Then there exists $\phi \in \Omega_c^{n_f(x)-2}(M)$, s.t.

$$\int_{\tilde{\partial}^2 W^u(x)} \phi \neq 0.$$

But by Proposition 3.5.4,

$$\int_{\tilde{\partial}^2 W^u(x)} \phi = \int_{W^u(x)} d^2\phi = 0,$$

a contradiction. $\square$

**Corollary 3.5.2.** *Let $\omega \in F_{Tf}^{[0,1],n_f(x)-1}$, one has*

$$\int_{W^u(x)} \exp(Tf) d_{Tf}\omega = \sum_{y\in \mathrm{Crit}(f), n_f(y)=n_f(x)-1} m(x,y)\int_{W^u(y)} \exp(Tf)\omega.$$

*In particular, the map $\mathcal{J}$ introduced in Section 3.3 is a chain map.*

*Proof.* By Theorem 1.3.1 and Lemma 3.3.5, for any $\epsilon > 0$, there exists $\phi \in \Omega_c^{n_f-1}(M)$ such that, for any $y \in \mathrm{Crit}(f)$ with $n_f(y) = n_f(x) - 1$,

$$\int_{W^u(x)} |\exp(Tf)d_{Tf}\omega - d\phi| < \epsilon, \int_{W^u(y)} |\exp(Tf)\omega - \phi| < \epsilon.$$

Now our Corollary follows from Proposition 3.5.4. □

We now turn to the proof of Proposition 3.5.4. We start with the following observation.

**Lemma 3.5.5.** *Let $(N, \partial N)$ be compact manifold with boundary. Moreover, assume that near the boundary $\partial N$, the manifold is of product type $(0, 1] \times \partial N$. Suppose that $f$ is a Morse function on $N - [1/2, 1] \times \partial N$. Then there exists a Morse function $\bar{f}$ on $N$, s.t. $\bar{f}|_{N-[1/4,1]\times\partial N} = f, \tilde{f}|_{[3/4,1]\times\partial N} = r$. Here $r$ is the standard coordinate on $(0, 1]$ factor.*

The proof is essentially the same as that of Theorem 2.5 in [43].

Recall from Section 3.3 that $\tilde{d}$ denote the distance function induced by the Agmon metric $|\nabla f|^2 g$. Let $\Phi^t$ denote the flow generated by $-\nabla f$. By reparameterization the results in Lemma 3.3.2 can be restated for $\Phi^t$ (and $\tilde{d}$). Namely, we have

$$|f(p) - f(q)| \leq \tilde{d}(p, q), \quad \forall p, q \in M. \tag{3.19}$$

and

$$\tilde{d}(\Phi^{t_1}(p), \Phi^{t_2}(p)) = |f(\Phi^{t_1}(p)) - f(\Phi^{t_2}(p))|. \tag{3.20}$$

Set (Cf. (3.13))

$$D = \sup_{y \in \mathrm{Crit}(f)} \tilde{d}(y, p_0) + 2 \sup_{y,z \in \mathrm{Crit}(f)} \tilde{d}(y, z).$$

**Lemma 3.5.6.** *For any fixed $x \in Crit(f)$ and any $\bar{D} > D$ and let $\tilde{B}_{\bar{D}}(x)$ be the ball centered at $x$ with radius $\bar{D}$ in the distance $\tilde{d}$ and $\tilde{B}^{\circ}_{\bar{D}}(x)$ the interior of $\tilde{B}_{\bar{D}}(x)$, Then for any $y, z \in \mathrm{Crit}(f)$, $W^u(y) \cap W^s(z) \subset \tilde{B}^{\circ}_{\bar{D}}(x)$. Moreover, if $p \notin \tilde{B}_{\bar{D}}(x)$ lies in the unstable manifold $W^u(x)$, then $\{\Phi^t(p) : t \geq 0\} \cap \tilde{B}_{\bar{D}}(x) = \emptyset$.*

*Proof.* Since $f$ is decreasing along the flow $\Phi^t$, by (3.20), for any $p \in W^u(y) \cap W^s(z)$,

$$\tilde{d}(y, p) = f(y) - f(p) \leq f(y) - f(z) = \tilde{d}(y, z).$$

Hence

$$\tilde{d}(x, p) \leq \tilde{d}(x, y) + \tilde{d}(y, p) \leq \tilde{d}(x, y) + \tilde{d}(y, z) \leq D.$$

Similarly, if $q \notin \tilde{B}_{\bar{D}}(x)$ lies in the unstable manifold $W^u(x)$, then for any $t \geq 0$,

$$\tilde{d}(x, \Phi^t(q)) = f(x) - f(\Phi^t(q)) \geq f(x) - f(q) = \tilde{d}(x, q) \geq \bar{D}$$

as desired. □

Now we are ready to prove Proposition 3.5.4

*Proof.* We reduce it to the compact case by a doubling construction and make use of Proposition 3.5.3.

For any $\phi \in \Omega_c^{n_f(x)}(M)$, let

$$\bar{D} := \sup_{p \in \mathrm{Crit} f \cup \mathrm{supp}(\phi)} \tilde{d}(p, p_0) + 2 \sup_{p,q \in \mathrm{Crit} f \cup \mathrm{supp}(\phi)} \tilde{d}(p, q),$$

we can find a compact submanifold $(N, \partial N)$ with boundary, such that $\tilde{B}_{\bar{D}}(x) \subset N^{\circ}$. Here $\mathrm{supp}(\phi)$ denotes the support of $\phi$, $N^{\circ}$ denote the interior of $N$. Thus, $\mathrm{supp}(\phi) \subset \tilde{B}^{\circ}_{\bar{D}}(x)$.

Now consider the double $(DN = N^+ \cup N^-, g_{DN})$ of $N$, $g_{DN}|_{\tilde{B}_{\bar{D}}(x)} = g$. By Lemma

3.5.5, we can find a Morse function $\bar{f}$ on $DN$, such that $\bar{f}|_{\tilde{B}_{\bar{D}}(x)} = f$. We may as well assume that $(DN, g_{DN}, \bar{f})$ satisfy Thom-Smale transversality condition. Then for any $y, z \in \mathrm{Crit}(\bar{f})$ with Morse index $n_{\bar{f}}(y) = n_{\bar{f}}(z) + 1$, let $m_{DN}(y, z)$ be the signed count of the number of flow lines in $W^u_{DN}(y) \cap W^s_{DN}(z)$, where $W^s_{DN}$ and $W^u_{DN}$ denote the stable and unstable manifolds with respect to $\bar{f}$ on $DN$.

Then, we note the following observations:

1. By Lemma 3.5.6 and its proof, if $z \in \tilde{B}_{\bar{D}}(x)$ is a critical point of $\bar{f}$ with $n_{\bar{f}}(z) = n_{\bar{f}}(x) - 1$, we have $m_{DN}(x, z) = m(x, z)$. Indeed, suppose $\gamma$ is a flow line on $DN$ connecting $x$ and $z$, and $\gamma$ is not contained in $\tilde{B}_{\bar{D}}(x)$. Let $w \in \gamma \cap \partial \tilde{B}_{\bar{D}}(x)$ be the place where $\gamma$ first meets $\partial \tilde{B}_{\bar{D}}(x)$. Then

   $$D \geq \tilde{d}(x, z) \geq f(x) - f(z) = \bar{f}(x) - \bar{f}(z) > \bar{f}(x) - \bar{f}(w) = f(x) - f(w) = \tilde{d}(x, w) = D,$$

   which is a contradiction. Here the strict inequality above follows from the fact that $\bar{f}$ decreases along its flow lines, and the second to the last equation follows from the fact that the part of flow lines of $\bar{f}$ inside $\tilde{B}_{\bar{D}}(x)$ coincides with flow lines of $f$ as $g_{DN}|_{\tilde{B}_{\bar{D}}(x)} = g, \bar{f}|_{\tilde{B}_{\bar{D}}(x)} = f$.

   As a result flow lines (if exist) connecting $x$ and $z$ in $DN$ must be contained in $\tilde{B}_{\bar{D}}(x)$. By Lemma 3.5.6, they are exactly flow lines connecting $x$ and $z$ in $M$. Therefore, $m_{DN}(x, z) = m(x, z)$.

2. If $z \notin \tilde{B}_{\bar{D}}(x)$ is a critical point of $\bar{f}$, and $W^s_{DN}(z) \cap W^u_{DN}(x) \neq \emptyset$, then $W^u_{DN}(z) \cap supp(\phi) = \emptyset$.

   This is because, let $\gamma$ be a flow line connecting $x$ and $z$ in $DN$, $w \in \gamma \cap \partial \tilde{B}_{\bar{D}}(x)$ be the first place where $\gamma$ meets $\partial \tilde{B}_{\bar{D}}(x)$. By (3.20), $\bar{f}(x) - \bar{f}(z) > \bar{f}(x) - \bar{f}(w) =$

$f(x) - f(w) = \tilde{d}(x, w) = \bar{D}$. Hence,

$$\bar{f}(z) < \inf_{p \in \operatorname{supp}(\phi)} \bar{f}(p). \tag{3.21}$$

Otherwise, there is $p \in \operatorname{supp}(\phi)$ such that $\bar{f}(z) \geq \bar{f}(p)$. Then by (3.19), $\bar{D} \geq \tilde{d}(x, p) \geq f(x) - f(p) = \bar{f}(x) - \bar{f}(p) \geq \bar{f}(x) - \bar{f}(z) > \bar{D}$. By (3.21), $W^u_{DN}(z) \cap supp(\phi) = \emptyset$.

As a result, by Proposition 3.5.3

$$\begin{aligned}
\int_{W^u(x)} d\phi = \int_{W^u_{DN}(x)} d\phi &= \sum_{z \in \operatorname{Crit}(\bar{f}), n_{\bar{f}}(z) = n_{\bar{f}}(x) - 1} m_{DN}(x, z) \int_{W^u_{DN}(z)} \phi \\
&= \sum_{y \in \operatorname{Crit}(f), n_f(y) = n_f(x) - 1} m_{DN}(x, y) \int_{W^u(y)} \phi \text{ (By Observation 2)} \\
&= \sum_{y \in \operatorname{Crit}(f), n_f(y) = n_f(x) - 1} m(x, y) \int_{W^u(y)} \phi \text{ (By Observation 1)},
\end{aligned}$$

as claimed. □

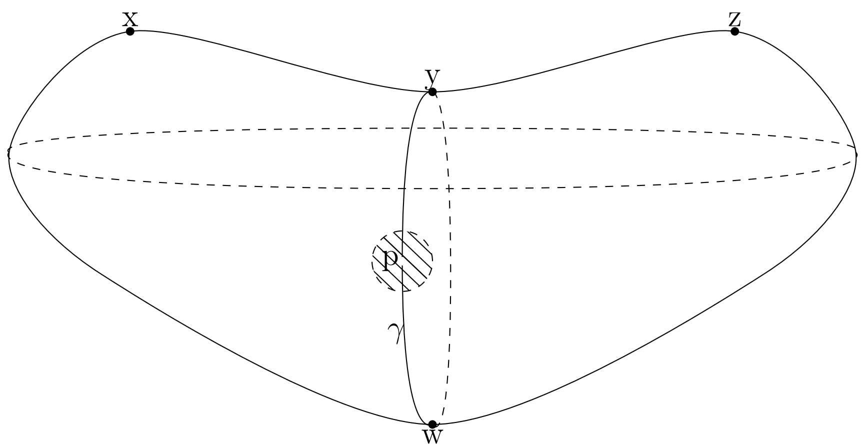
**Remark 3.5.7.** *Here we have made essential use of the fact that $|\nabla f|$ has a positive lower bound outside some compact set $K_0$. Indeed, in this case, $(M, |\nabla f|^2 g)$ is complete and hence, $\tilde{B}_r(p)$ is compact for all $r > 0, p \in M$. Therefore one can always find a compact manifold with boundary $N$ containing $\tilde{B}_{\bar{D}}(x)$. Moreover, by our choice of $\bar{D}$, for all $q \in (M - \tilde{B}_{\bar{D}}(x)) \cap W^u(x)$, $f(q) < \inf_{q' \in \operatorname{supp}(\phi) \cup \operatorname{Crit}(f)} f(q')$. Therefore, since $f$ is decreasing along the flow, once a flow line escapes $\tilde{B}_{\bar{D}}(x)$, it never flows back to* $\operatorname{supp}(\phi) \cup \operatorname{Crit}(f)$. *Consequently, we have Lemma 3.5.6 , Observation 1 and 2.*

### A Counterexample

To close out this subsection, we present a counterexample provided by Shu Shen which showes that, if one drops the condition that $|\nabla f|$ has a positive lower bound near infinity, the conclusion $\tilde{\partial}^2 = 0$ can fail.

Consider the following heart shaped topological sphere $S$ with $f$ being the height function. Then we have four critical points $x, y, z, w$ as indicated below. Let $\gamma$ be a flow line connecting $y$ and $w$, and remove a point $p$ on $\gamma$. Make a conformal change of metric near the point $p$ so that $S - \{p\}$ is complete under this new metric. Now one can check that $|\nabla f(q)| \to 0$, as $q \to p$. On the other hand, since the flow line is invariant under the conformal change of metric, $\gamma - \{p\}$ is still a (broken) flow line. And in this case, $\tilde{\partial}^2 x = w$, which is nonzero.

In our previous arguments, the fact that $|\nabla f|$ has a positive lower bounded near the infinity play a crucial role. See Remark 3.5.7 above.




**Remark 3.5.8.** *We would like to thank Shu Shen for providing this interesting example.*

### 3.5.4 Isomorphism of $H^*(C^\bullet(W^u), \tilde{\partial}')$ and $H^*_{dR}(M, U_c)$

For simplicity, we assume that $f$ is a self-indexed Morse function, i.e., if $x$ is a critical point of $f$ with Morse index $i$, we require $f(x) = i$.

Let $V_i = f^{-1}(-\infty, i + \frac{1}{2}]$, $0 \leq i \leq n$.

Recall that we assume in a neighborhood $U_x$ of critical points $x$ of $f$, we have coordinate system $z = (z_1, ..., z_n)$, such that

$$f = f(x) - z_1^2 - ... - z_{n_f(x)}^2 + z_{n_f(x)+1}^2 + ... + z_n^2,$$

$$g = dz_1^2 + ... + dz_n^2,$$

Moreover $U_x$ is an Euclidean open ball around $x$ with radius 1. Also, these open balls are disjoint.

We have the following observation:

**Lemma 3.5.9.** *$V_0$ can be written as disjoint union of $\cup_{x \in Crit(f), n_f(x)=0} \tilde{U}_x$ and $V$, where $V$ is some open subset diffeomorphic to $U_c$, $\tilde{U}_x$ is an Euclidean ball around $x$ with radius $\frac{1}{2}$. Also, $V_n$ is diffeomorphic to $M$.*

*Proof.* Let $X_f := \frac{\nabla f}{|\nabla f|^2}$, $\Phi^t$ be the flow generated by $X_f$. Then we have

$$\left(\Phi^{c+\frac{1}{2}}(U_c)\right) \cap \left(\cup_{x \in Crit(f), n_f(x)=0} \tilde{U}_x\right) = \emptyset.$$

This is because:

- If $f(p) \leq c - \frac{1}{2}$, then $f(\Phi^{c+\frac{1}{2}}(p)) < 0$. Hence $\Phi^{c+\frac{1}{2}}(p) \notin \cup_{x \in Crit(f), n_f(x)=0} \tilde{U}_x$.
- If $c - \frac{1}{2} \leq f(p) < c$, and if $\Phi^{c+\frac{1}{2}}(p) \in \tilde{U}_x$ for some $x \in Crit(f)$ with Morse index $n_f(x) = 0$. Then $\Phi^{c+\frac{1}{2}}(p) \in W^s(x)$, which implies $p \in W^s(x)$. But this is impossible since $f(p) < -c < 0 = f(x)$.

We can similarly prove that $V_n$ is diffeomorphic to $M$. $\square$

Let $C_*(V_i, U_c)$ be the complex of relative singular chains. Then we have

$$C_*(V_n, U_c) \supset C_*(V_{n-1}, U_c) \supset \cdots C_*(V_0, U_c).$$

By Lemma 3.5.9 and a spectral sequence argument similar to the proof of Theorem 1.6 in [20], one can show that

$$H_*(C^\bullet(W^u), \tilde{\partial}) \simeq H_*(M, U_c).$$

Thus, it follows from the universal coefficient theorem that

$$H^*(C^\bullet(W^u), \tilde{\partial}') \simeq H^*_{dR}(M, U_c).$$

### 3.5.5 Isomorphism of $H^*_{(2)}(M, d_{Tf})$ and $H^*(C^\bullet(W^u), \tilde{\partial}')$

We will first show that the chain map $\mathcal{J} : (F^{[0,1],*}_{Tf}, d_{Tf}) \mapsto C^*((W^u)', \tilde{\partial}')$ defined in Section 3.3 is in fact an isomorphism when $T$ is sufficiently large. Hence $\mathcal{J}$ induces an isomorphism between $H^*_{(2)}(M, d_{Tf})$ and $H^*(C^\bullet(W^u), \partial)$ in that case.

More precisely the arguments follow those in Chapter 6 of [27], with a necessary modification, and we will only indicate the modification here. The basic idea is to construct an explicit map which approximate the inverse of $\mathcal{J}$ (up to constant multiple) as $T \to \infty$. Therefore, there exists $T_6 > T_0$, such that $\mathcal{J}$ is an isomorphism whenever $T > T_6$. (We point out that the explicit description of $T_6$ is more involved than $T_0$.)

In fact, the modification we need is a more refined estimate in Theorem 6.7 of [27]. Namely, we have

$$|\mathcal{P}\tau_{x,T} - \tau_{x,T}| \leq C \exp(-a'T\sqrt{\rho^2 + 1})\|\tau_{x,T}\|_{L^2}, \tag{3.22}$$

where $\mathcal{P}$ is the orthogonal projection from $L^2\Lambda(M)$ to $F^{[0,1],*}$, and $C$, $a' < a$ are positive constants.

Here $\tau_{x,T}$ is defined as follows (and the explicit map from $C^*((W^u)', \tilde{\partial}')$ to $(F^{[0,1],*}_{Tf}, d_{Tf})$

assigns a (normalizing) multiple of $\mathcal{P}\tau_{x,T}$ to $W^u(x)^*$). Notice that in Section 3.4, we require that in a neighborhoof $U$ of $x$, the metric and Morse function is of the form (3.15). Let $\alpha_x$ be a bump function whose support is contained in $U$ and $\alpha_x \equiv 1$ in a neighborhood $V$ of $x$, and set

$$\tau_{x,T} = \alpha_x \exp(-T^2|z|^2) dz_1 \wedge \cdots \wedge dz_{n_f(x)}.$$

Then $\Box_{Tf}\tau_{x,T} = 0$ in $V$ and $M - U$.

To obtain the estimate (3.22), pick a bump function $\eta$ with compact support , such that $\eta \equiv 1$ on $K$. Then our Agmon estimate yields

$$|(1-\eta)(\mathcal{P}\tau_{x,T} - \tau_{x,T})| \leq C \exp(-aT\rho)\|\tau_{x,T}\|_{L^2}.$$

On the other hand, the estimate

$$|\eta(\mathcal{P}\tau_{x,T} - \tau_{x,T})| \leq C \exp(-cT)\|\tau_{x,T}\|_{L^2}$$

follows from exactly the same argument in the proof of Theorem 6.7 of [27].

Now it remains to prove that when $T \in (T_0, T_6]$, $H^*_{(2)}(M, d_{Tf})$ and $H^*(C^\bullet(W^u), \partial)$ are still isomorphic.

We only present the proof for the case when $(M, g, f)$ is strongly tame, the case of well tame being exactly the same except notationally. In this case, we have $T_0 = 0$. The idea is to show that if $S > 0$, then for any $T \in [7/8S, S]$, $H^*_{(2)}(M, d_{Tf})$ and $H^*_{(2)}(M, d_{Sf})$ are isomorphic. Hence $H^*_{(2)}(M, d_{Tf})$ is independent of $T \in (0, \infty)$, which finishes the proof of isomorphism of $H^*_{(2)}(M, d_{Tf})$ and $H^*(C^\bullet(W^u), \partial)$.

For simplicity, we prove that $H^*_{(2)}(M, d_{7f})$ and $H^*_{(2)}(M, d_{8f})$ are isomorphic, the general case being similar.

Thus fix coefficients $a = \frac{63}{64}$, $b = \frac{127}{128}$ in Lemma 3.1.1 and Theorem 1.3.1.

Define $M_f : (F_{8f}^{*,[0,1]}, d_{8f}) \mapsto (\Omega_{(2)}^*(M), d_{7f})$; $\forall w \in F_{8f}^{*,[0,1]}$, $M_f(w) = \exp(f)w$. Similarly $M_{-f} : (F_{7f}^{*,[0,1]}, d_{7f}) \mapsto (\Omega_{(2)}^*(M), d_{8f})$; $\forall w \in F_{7f}^{*,[0,1]}$, $M_{-f}(w) = \exp(-f)w$.

Clearly these are chain maps once we check that $M_f$ and $M_{-f}$ are well defined. To this end, let's verify that $|f(p)| \leq \sup_{q\in K} |f(q)| + \frac{1}{bT}\rho_T(p)$. Indeed, let $\gamma : [0, \rho_T(p)]$ be a normal minimal geodesic connecting $K$ and $p$, in the metric $\tilde{g}_T$. Then

$$|\frac{d}{dt} f \circ \gamma(t)| = | < \tilde{\nabla} f, \gamma' >_{\tilde{g}_T} | \leq \frac{1}{bT}.$$

Now the $L^2$ bound of $M_f(w)$ (resp. $M_{-f}(w)$) follows by Theorem 1.3.1 and the standard volume comparison. Hence $M_f$ induces a homomorphism (still denote it by $M_f$) from $H_{(2)}^*(M, d_{8f})$ to $H_{(2)}^*(M, d_{7f})$.

Our next step is to show that $M_f$ is injective. Suppose we have $w \in \ker(\Box_{8f})$, s.t. $M_f w$ is exact, which means that we can find $\alpha \in Im(\delta_{7f})$, s.t $\exp(f)w = d_{7f}\alpha (= (d_{7f} + \delta_{7f})\alpha)$.

Thus

$$\Box_{7f}\alpha = (d_{7f} + \delta_{7f}) \exp(f)w = \exp(f) d_{8f} w + \exp(2f)\delta_{6f} w$$

$$= 0 + \exp(2f)(\delta_{8f} w - \iota_{2f} w) = -\exp(2f)\iota_{2f} w.$$

By Lemma 3.5.2, $|\alpha| \leq C \exp(-1/3\rho_7)$. Consequently, $\exp(-f)\alpha \in L^2\Lambda^*(M)$, and $w = d_{8f} \exp(-f)\alpha$ is exact.

As a result, $M_f$ is injective. Similarly, $M_{-f}$ is also injective. Therefore, $H_{(2)}^*(M, d_{8f})$ and $H_{(2)}^*(M, d_{7f})$ are isomorphic.

# Chapter 4

# Heat kernel Expansion and Local Index Theorem

In this chapter, we study the asymptotic expansion of the heat kernel associated to the Witten Laplacian which is essentially a Schrödinger operator. Not much is known previously about asymptotic expansions of heat kernels and heat traces of Schrödinger operators on non-compact spaces. Even for cases as simple as $\mathbb{C}^n$ with polynomial potentials, it is already very complicated. Motivated by path integral formulation of the heat kernel, we found that a parabolic distance, introduced in Li-Yau's famous work on the parabolic Harnack estimate, provides a much simpler and satisfying approach. Using the Li-Yau parabolic distance, we derive a pointwise asymptotic expansion of the heat kernel for the Witten Laplacian with strong remainder estimate. When the deformation parameter of Witten deformation and time parameter are coupled, we derive an asymptotic expansion of trace of heat kernel for small-time $t$, and obtain a local index theorem. In the special case of $\mathbb{C}^n$ with a quasi-homogeneous polynomial, the corresponding index formula reduces to the Milnor number of the polynomial.

## 4.1 Weak Weyl Law

In this section we will show that the polynomial tame condition implies that $\exp(-t\Box_{Tf})$ is of trace class. This is achieved by proving a weak Weyl law which shows that the eigenvalues of the Witten Laplacian grows polynomially. The Agmon estimate developped in [25] plays a crucial role here.

### 4.1.1 Review of Hodge Theory for Witten Laplacian

For any $T > 0$, let

$$d_{Tf} := d + Tdf\wedge : \Omega^*(M) \mapsto \Omega^{*+1}(M)$$

be the so-called Witten deformation of de Rham operator $d$. As usual, the metric $g$ induces a canonical metric (still denote it by $g$) on $\Lambda^*(M)$, which then defines an inner product $(\cdot,\cdot)_{L^2}$ on $\Omega_c^*(M)$:

$$(\phi,\psi)_{L^2} = \int_M (\phi,\psi)_g dvol, \phi,\psi \in \Omega_c^*(M).$$

Let $L^2\Lambda^*(M)$ be the completion of $\Omega_c^*(M)$ with respect to $\|\cdot\|_{L^2}$, and $L^2(M) := L^2\Lambda^0(M)$.

Then $d_{Tf}$ is an unbounded operator on $L^2\Lambda^*(M)$ with domain $\Omega_c^*(M)$. Also, it has a formal adjoint operator $\delta_{Tf}$, with $\mathrm{Dom}(\delta_{Tf}) = \Omega_c^*(M)$, such that

$$(d_{Tf}\phi,\psi)_{L^2} = (\phi,\delta_{Tf}\psi)_{L^2}, \phi,\psi \in \Omega_c^*(M).$$

Set $\Delta_{H,Tf} = (d_{Tf} + \delta_{Tf})^2$, and we denote the Friedrichs extension of $\Delta_{H,Tf}$ by $\Box_{Tf}$. If $(M,g)$ is complete then $\Delta_{H,Tf}$ is essentially self-adjoint (and hence $\Box_{Tf}$ is the unique self-adjoint extension). In [25] We proved that when $(M, g, f)$ is tame,

$$L^2\Lambda^*(M) = \ker \square_{Tf} \oplus \mathrm{Im}\bar{d}_{Tf} \oplus \mathrm{Im}\bar{\delta}_{Tf}, \tag{4.1}$$

where $\bar{d}_{Tf}$ and $\bar{\delta}_{Tf}$ are the graph extensions of $d_{Tf}$ and $\delta_{Tf}$ respectively.

Setting $\Omega^*_{(2)}(M, Tf) := \mathrm{Dom}(\bar{d}_{Tf}) \cap \Omega^*(M)$, we have a chain complex

$$\cdots \xrightarrow{d_{Tf}} \Omega^*_{(2)}(M, Tf) \xrightarrow{d_{Tf}} \Omega^{*+1}_{(2)}(M, Tf) \xrightarrow{d_{Tf}} \cdots .$$

Let $H^*_{(2)}(M, d_{Tf})$ denote the cohomology of this complex. In [25], we have shown that $H^*_{(2)}(M, d_{Tf}) \cong \ker \square_{Tf}$, provided $(M, g, f)$ is well tame and $T$ is large enough. Note that the notion of well tame [25] is strictly weaker than that of regular tame.

Finally, we note the following well known

**Proposition 4.1.1.** *Denote $L_f = \nabla^2_{e_i,e_j} f[e^i\wedge, \iota_{e_j}]$ locally, where $\{e_i\}$ is a local frame on $TM$ and $\{e^i\}$ is the dual frame on $T^*M$. Then the Witten Laplacian $\Delta_{H,Tf}$ has the following expression:*

$$\Delta_{H,Tf} = \Delta - TL_f + T^2|\nabla f|^2. \tag{4.2}$$

*Here $\Delta$ denotes the Hodge Laplacian.*

### 4.1.2 Weak Weyl Law for Witten Laplacian

Let $(M, g, f)$ be $\alpha$-polynomial tame defined in the previous section. Then, $(M, g, f)$ is regular tame and there is some constant $C$, such that for all $\lambda \geq 0$,

$$\int_{\{p\in M:|\nabla f|^2(p)\leq\lambda\}} (\lambda - |\nabla f|^2)^{n/2} dvol_M \leq C\lambda^\alpha.$$

This has the following immediate consequences.

**Lemma 4.1.2.** *Let* $K_\lambda := \{p \in M : |\nabla f|^2(p) < \lambda\}$, *then*

$$\mathrm{Vol}(K_\lambda) \leq C\lambda^{\alpha - \frac{n}{2}}.$$

*Furthermore, for any* $k \geq 0$, *there is a constant* $C_k$ *depending only on* $k$ *and the tameness condition such that*

$$\int_M \exp(-|\nabla f|^2)|\nabla f|^k dvol \leq C_k.$$

*Proof.* We have

$$\lambda^{\frac{n}{2}} \mathrm{Vol}(K_\lambda) \leq \int_{K_{2\lambda}} (2\lambda - |\nabla f|^2)^{\frac{n}{2}} \leq C\lambda^\alpha.$$

To prove the second estimate, we notice that

$$\begin{aligned}
\int_M \exp(-|\nabla f|^2)|\nabla f|^k dvol &= \sum_{l=0}^{\infty} \int_{K_{l+1}-K_l} \exp(-|\nabla f|^2)|\nabla f|^{\frac{k}{2}} dvol \\
&\leq \sum_{l=0}^{\infty} e^{-l}(l+1)^{\frac{k}{2}} \mathrm{Vol}(K_{l+1} - K_l) \\
&\leq C \sum_{l=0}^{\infty} e^{-l}(l+1)^{\frac{k}{2}+\alpha-\frac{n}{2}} = C_k < \infty,
\end{aligned}$$

as desired. □

Note that, in particular, if $\alpha = n/2$, then $M$ must have finite volume.

We now turn our attention to the growth of eigenvalues of the Witten Laplacian. First, by refining the argument of Theorem 1.1 in [25], we have the following exponential decay estimate for eigenforms.

**Proposition 4.1.3.** *Let* $(M, g, f)$ *be strongly tame, and* $\omega \in \mathrm{Dom}(\Box_f)$ *be an eigenform* *of $\Box_f$ with eigenvalue $\lambda \geq 1$. Then*

$$|\omega(p)| \leq C(M,a)(1+\delta_f)\lambda \exp(-a\rho_{5\lambda}(p))\|\omega\|_{L^2}, \tag{4.3}$$

*for any $a \in (0,1)$ whenever $p \notin K_{3\lambda}$. Here $\rho_\lambda$ is the Agmon distance induced by Agmon metric $g_\lambda := (|\nabla f|^2 - \lambda)_+ g$, with $(|\nabla f|^2 - \lambda)_+$ denoting the nonnegative part, $\delta_f$ is a positive number such that $\delta_f > \sup_{p\in M} \frac{|Hess(f)|}{(|\nabla f|+1)^2}$, and $C$ is a constant depending on the curvature bounds and $a$.*

*Proof.* Indeed, in the proof of Lemma 3.1 in [25], we let $K = K_\lambda$, $L = K_{2\lambda} \cap \{p : \rho_\lambda(p) < 2\}$, $\varphi_{k,j} = \mu_k \exp(b\rho_{T,j})$, $\lambda_T = a^2(|\nabla f|^2 - \lambda)_+$ for some $a \in (0,1)$, $w = 0$ instead. Here for $k$ large, $\mu_k$ is a bump function that satisfies

(a) $|\nabla \mu_k|^2 \leq C(1+\delta_f)\lambda$ in $L - K$.

(b) $|\nabla \mu_k|^2(p) \leq 2(|\nabla f|^2 - \lambda)_+$ when $p \in \{q \in M : \rho_\lambda(q) \in (k, k+1)\}$.

(c) $\mu_k(p) = 1$ when $p \in M - L \cap \{q \in M : \rho_\lambda(q) < k\}$.

(d) $\mu_k(p) = 0$ when $p \in K \cup \{q \in M : \rho_\lambda(q) > k+1\}$.

Indeed, it suffices to verify (a), to this end, we show that

$$D := \mathrm{dist}(K, M-L) \geq \frac{c}{(\delta_f+1)\lambda} \tag{4.4}$$

with respect to metric $g$. Let $\gamma : [0, D] \to M$ be a shortest normal geodesic connecting $K$ and $M - L$ with respect to metric $g$, such that $\gamma(0) \in \partial K$, $\gamma(D) \in \partial L$. We must have $\gamma \subset \bar{L}$. Moreover, if $\rho_\lambda \circ \gamma(D) = 2$, then since $|\nabla \rho_\lambda|^2 = (|\nabla f|^2 - \lambda)_+ \leq \lambda$ on $L$, we must have $D \geq \frac{2}{\lambda}$. If $\rho_\lambda \circ \gamma(D) < 2$, we must have $|\nabla f| \circ \gamma(D) = \sqrt{2\lambda}$. Since $|\nabla|\nabla f|| \leq |Hessf| \leq \delta_f(1 + |\nabla f|^2) \leq 3\delta_f\lambda$, we must have $D \geq \frac{1}{3\delta_f\sqrt{\lambda}} \geq \frac{1}{3\delta_f\lambda}$. Consequently, (3.13) is true.

Now we still have (see the proof of Lemma 3.1 in [25]):

$$\int_{M-L} |u|^2(|\nabla f|^2-\lambda)_+ \exp\left(2b\rho_{\lambda,j}\right) \mathrm{dvol} \le C_1(b)(1+\delta_f)\lambda \int_{L-K} |u|^2 \exp\left(2b\rho_{\lambda,j}\right) \mathrm{dvol}$$
$$\le C_2(b)(1+\delta_f)\lambda \int_{L-K} |u|^2 \mathrm{dvol},$$

where $\rho_{\lambda,j} = \min \rho_{\lambda,j}$ for all $j \in \mathbb{Z}^+$.

Since on $K_{3\lambda}$, $(|\nabla f|^2-\lambda)^+ \ge 5\lambda$, one has

$$\int_{M-K_{6\lambda}} |u|^2 \exp\left(2b\rho_{\lambda,j}\right) \mathrm{dvol} \le C_3(b)(1+\delta_f) \int_{L-K} |u|^2 \mathrm{dvol}.$$

Proceed as in [25], one can see that when $p \in M - K_{3\lambda}$, (4.3) is true.

$\square$

With the help of Proposition 4.1.3, we now deduce a weak version of Weyl's law:

**Proposition 4.1.4.** *If $(M,g,f)$ is $\alpha$-polynomial tame, then the spectrum of $\Box_f$ has polynomial growth. More precisely, there exist constants $\delta > 0$ and $C > 0$, such that $\lambda_k(\Box_f) \ge Ck^\delta$, where $\lambda_k(\Box_f)$ denotes the $k$-th eigenvalue of $\Box_f$ (counted with multiplicity). Consequently,* $\exp(-t\Box_{Tf})$ *is of trace class for all $T > 0, t > 0$.*

*Proof.* Let $E(\lambda)$ be the number of eigenvalues not exceeding $\lambda$, and $u$ be an eigenform with eigenvalue $\lambda_0 \le \lambda$. We normalize $u$ so that $\|u\|_{L^2} = 1$ .

By Proposition 4.1.3,

$$|u(p)| \le C\lambda \exp(-a\rho_\lambda(p)).$$

We claim that there exists $n_0 > 1$ independent of $\lambda \ge 1$ and $u$, such that

$$\int_{M-K_{n_0\lambda}} |u|^2 dvol < \frac{1}{2}. \tag{4.5}$$

To prove the claim we first estimate the Agmon distance. Thus, for $p \in K_{(k+1)\lambda} - K_{k\lambda}$, let $\gamma : [0, l] \mapsto M$ be a minimal curve in the Agmon metric $g_\lambda$ connecting $p$ and $K_{k\lambda}$; moreover, $|\gamma'(s)| = 1$ with respect to the metric $g$. Then we may as well assume that $\gamma \subset K_{(k+1)\lambda}$; otherwise, we can find $l_0 \in [0, l]$, such that $\gamma|_{[l_0,l]} \subset K_{(k+1)\lambda}$ and we can take $p = \gamma(l_0)$. Hence by the tameness condition, there exists $c > 0$, s.t.

$$\frac{d}{dt}(|\nabla f|^2 \circ \gamma(t)) \leq c|\nabla f|^{\kappa+2} \leq c((k+1)\lambda)^{\frac{\kappa+2}{2}}.$$

It follows by integrating that $l \geq \frac{|\nabla f|^2(p) - k\lambda}{((k+1)\lambda)^{\frac{\kappa+2}{2}}}$. In particular, if $L$ is the $g_\lambda$-length of $\gamma$ such that that $|\nabla f|^2(p) = (k+1)\lambda$, then for some $c' > 0$

$$L = \int_0^l (|\nabla f|^2 - \lambda)^{\frac{1}{2}} \circ \gamma(t) dt \geq \frac{(k-1)^{\frac{1}{2}} \lambda^{\frac{1-\kappa}{2}}}{(k+1)^{\frac{\kappa+2}{2}}} \geq \frac{c' \lambda^{\frac{1-\kappa}{2}}}{k^{\frac{\kappa+1}{2}}}.$$

Hence, if $x \in K_{(k+1)\lambda} - K_{k\lambda}$, then (say $k \geq 3$)

$$\rho_\lambda(x) \geq \sum_{i=2}^{k-1} \frac{c' \lambda^{\frac{1-\kappa}{2}}}{i^{\frac{\kappa+1}{2}}} \geq c'' \lambda^{\frac{1-\kappa}{2}} k^{\frac{1-\kappa}{2}}$$

for some constant $c'' > 0$.

Therefore, assuming $n_0 > 5$,

$$\begin{aligned}\int_{M-K_{n_0\lambda}} |u|^2 dvol &= \sum_{k=n_0}^{\infty} \int_{K_{(k+1)\lambda}-K_{k\lambda}} |u|^2 dvol \\ &\leq \sum_{k=n_0}^{\infty} \int_{K_{(k+1)\lambda}-K_{k\lambda}} \lambda C e^{-a\rho_\lambda} dvol \\ &\leq \sum_{k=n_0}^{\infty} C\lambda e^{-ac''\lambda^{\frac{1-\kappa}{2}} k^{\frac{1-\kappa}{2}}} \operatorname{Vol}(K_{(k+1)\lambda}) \\ &\leq \sum_{k=n_0}^{\infty} C_1 \lambda e^{-ac''\lambda^{\frac{1-\kappa}{2}} k^{\frac{1-\kappa}{2}}} ((k+1)\lambda)^{\alpha-\frac{n}{2}} \\ &\leq \sum_{k=n_0}^{\infty} C_1 C' e^{-\frac{1}{2}ac'\lambda^{\frac{1-\kappa}{2}} k^{\frac{1-\kappa}{2}}} \leq \sum_{k=n_0}^{\infty} C_1 C' e^{-\frac{1}{2}ac''k^{\frac{1-\kappa}{2}}}\end{aligned}$$

for $\lambda \geq 1$. Here $C' = \max_{\eta>0} \eta^{\frac{2\alpha-n+2}{1-\kappa}} e^{-\frac{1}{2}ac''\eta}$. Clearly there is some $n_0$ independing $\lambda$ such that the last term in the inequality above is less than $1/2$, which finishes the proof of the claim.

Let $N(\epsilon, \lambda)$ be the minimal number of elements in an $\epsilon$-dense subset of $K_{n_0\lambda}$. Then by the volume comparison, $N(\epsilon, \lambda) \leq C_2 \frac{\operatorname{Vol}(K_{n_0\lambda})}{\epsilon^n}$. We now follow the argument in the proof of Theorem 5.8 of [44] to show that $E(\lambda) \leq N(\epsilon, \lambda)$ for suitable $\epsilon$. Indeed, if $E(\lambda) > N(\epsilon, \lambda)$, then there exists $u \in E(\lambda)$ with unit $L^2$ norm which vanishes on an $\epsilon$-dense subset of $K_{n_0\lambda}$. By using the elliptic estimate as in [44] one deduces

$$\sup_{K_{n_0\lambda}} |u| \leq \epsilon C_k (1 + \lambda^k)$$

for any $2k > \frac{n}{2}+1$. But this is clearly impossible if we take $\epsilon^{-1} := 2C_k(1+\lambda^k)\operatorname{Vol}(K_{n_o\lambda})^{1/2}$, as

$$\int_{K_{n_0\lambda}} |u|^2 dvol > 1/2.$$

.

As a result, if we choose the minimal $k$, s.t. $2k > \frac{n}{2} + 1$, then by Lemma 4.1.2,

$$E(\lambda) \leq N(\epsilon, \lambda) \leq C_1 \frac{\mathrm{Vol}(K_{n_0\lambda})}{\epsilon^n} \leq C\lambda^{\frac{n}{2}\alpha+\alpha+n}.$$

The rest of the proposition follows. □

**Remark 4.1.5.** *The $\alpha$-polynomial tame condition is a technical one for the usual heat kernel approach to local index theorems. For example, on $\mathbb{R}$ consider $f = |x|\ln|x|$ outside $|x| \leq e$. Let $\lambda_k$ be the $k$-th eigenvalue of $\Box_f$. Then by Weyl's law (Cf. [29]), $\lambda_k \lesssim \sqrt{\ln(k)}$. For such slowly growing eigenvalue distributions, it is unreasonable to consider the limit $\lim_{t\to 0} Tr_s(\exp(-t\Box_f))$. On the other hand, this assumption is not essential if one is only interested in an index formula. This issue will be elaborated in a separate paper when we discuss the Dirac/Callias type operators.*

Thus, assuming the $\alpha$-polynomial tame condition, $\exp(-t\Box_{Tf})$ is of trace class. It follows that

$$h(t,T) = \mathrm{Tr}_s(\exp(-t\Box_{Tf})) \tag{4.6}$$

is independent of $t$. Moreover, as $t \to \infty$, $h(t,T) \to \chi(M, d_{Tf})$, where

$$\chi(M, d_{Tf}) = \sum_{i=0}^{n} (-1)^i b_i(T), b_i = \dim(H^i_{(2)}(M, d_{Tf})).$$

Now by Theorem 1.3 in [25], $h(t,T)$ is independent of $T > 0$. As a result, $h(t,T)$ is independent of both $t > 0$ and $T > 0$.

## 4.2 Construction of Parametrix

In this section, we extend the parametrix construction of the heat kernel to the Witten deformation. The case of Euclidean space is treated in [33].

Fix $x \in M$, and let $d(y, x)$ be the distance function. Let $\tau > 0$ be the injectivity radius of $M$. Then for $y \in B_\tau(x)$, define

$$\mathcal{E}_0(t, x, y) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp(-d^2(x, y)/4t). \tag{4.7}$$

For simplicity, we denote $V_T = T^2|\nabla f|^2$ and $V = |\nabla f|^2$. Suppose $\gamma$ is the normal geodesic connecting $x$ and $y$, and $r_x(y) = d(x, y)$. Set

$$h_T(x, y) = \frac{1}{r_x(y)} \int_0^{r_x(y)} V_T(\gamma(s))ds = T^2 h(x, y), \quad h(x, y) = \frac{1}{r_x(y)} \int_0^{r_x(y)} V(\gamma(s))ds. \tag{4.8}$$

We define

$$\mathcal{E}_{1,T}(t, x, y) = \exp(-t\, h_T(x, y)). \tag{4.9}$$

Then direct computation gives us the following formulas (the first two are well known).

**Proposition 4.2.1.** *For $y \in B_\tau(x)$ in the normal coordinates near $x$, we have*

$$\nabla \mathcal{E}_0 = -\frac{\mathcal{E}_0}{2t} r \nabla r, \quad (\frac{\partial}{\partial t} + \Delta)\mathcal{E}_0 = \frac{\mathcal{E}_0}{4tG} \nabla_{r\nabla r} G.$$

$$\nabla_{r\nabla r} h_T(x, y) + h_T(x, y) - V_T(y) = 0.$$

*Here $G = \det(g_{ij})$ and derivatives are taken with respect to $y$.*

Let $p_i : M \times M \mapsto M$ be the projection of i-th factor of $M \times M$ to $M$, $i = 1, 2$. We define the vector bundle $E \to M \times M$ to be $E = (p_1)^*(\Lambda^*(M)) \otimes (p_2)^*(\Lambda^*(M))$. Let $s(t, x, y) = \sum_{i=0}^k t^i \Theta_i(x, y)$, where $\Theta_i(x, y) \in \Gamma(E)$. Since $y$ is within the injectivity radius of $x$, we use parallel transport along radius geodesics to identify $\Lambda_y^*(M)$ with $\Lambda_x^*(M)$. In this way, $\Theta_i(x, y) \in \Gamma(E)$ is identified with an endomorphismm of $\Lambda^*(M)$ using the metric. Again by a straightforward computation and using Proposition 4.2.1, we have

**Proposition 4.2.2.**

$$
\begin{aligned}
&(\frac{\partial}{\partial t}+\square_{Tf})(\mathcal{E}_0\mathcal{E}_{1,T}s)\\
&=\mathcal{E}_0\mathcal{E}_{1,T}\Big\{\sum_{j=-1}^{k-1}[(j+1+\frac{1}{4G}\nabla_{r\nabla r}G)\Theta_{j+1}+\nabla_{r\nabla r}\Theta_{j+1}+\Delta\Theta_j-TL_f\Theta_j]t^j\\
&+[\Delta\Theta_k-TL_f\Theta_k]t^k+\sum_{j=1}^{k+1}[-\Delta h_T\Theta_{j-1}+2\nabla_{\nabla h_T}\Theta_{j-1}]t^j\\
&+\sum_{j=2}^{k+2}[-|\nabla h_T|^2\Theta_{j-2}]t^j\Big\},
\end{aligned}
\tag{4.10}
$$

*where the derivatives are taken with respect to $y$.*

Now we can follow the standard procedure to find suitable $\Theta_j=\Theta_{T,j}$ with $\Theta_{T,0}(x,x)=\mathrm{Id}$, $j=0,1,...,k$, such that

$$
(\frac{\partial}{\partial t}+\square_{Tf})(\mathcal{E}_0\mathcal{E}_1 s)=t^kR_{k,T}(t,x,y),
\tag{4.11}
$$

where $R_{k,T}(t,x,y)$ is $C^0$ in $t\in[0,\infty)$. This amounts to solving ODEs inductively.

For $j=-1$, we have $\frac{d}{dr}(G^{1/4}\Theta_{T,0})=0$. Together with the initial condition $\Theta_{T,0}(x,x)=\mathrm{Id}$, one has $\Theta_{T,0}=G^{-1/4}\,\mathrm{Id}$.

For $j=0$, we have $\frac{d}{dr}(rG^{1/4}\Theta_{T,1})=G^{1/4}(TL_f-\Delta)\Theta_{T,0}$; hence we can solve $\Theta_1$ explicitly in terms of $\Theta_{T,0}$, by integrating along the geodesic.

Similarly, for $1\le j\le k-1$, $\Theta_{T,j+1}$ can be solved recursively from the equation

$$
\begin{aligned}
\frac{d}{dr}(r^{j+1}G^{1/4}\Theta_{T,j+1})=-r^jG^{1/4}(&\Delta\Theta_{T,j}-TL_f\Theta_{T,j}-\Delta h_T\Theta_{T,j-1}\\
&+2\nabla_{\nabla h_T}\Theta_{T,j-1}-|\nabla h_T|^2\Theta_{T,j-2}).
\end{aligned}
$$

With these choices for $\Theta_{T,j}$'s, we obtain (4.11), where

$$R_{k,T} = \mathcal{E}_0\mathcal{E}_1\Big\{[\Delta\Theta_{T,k} - TL_f\Theta_{T,k} - \Delta h_T\Theta_{T,k-1} + 2\nabla_{\nabla h_T}\Theta_{T,k-1} - |\nabla h_T|^2\Theta_{T,k-2}] + [-\Delta h_T\Theta_{T,k} + 2\nabla_{\nabla h_T}\Theta_{T,k} - |\nabla h_T|^2\Theta_{T,k-1}]t + [-|\nabla h_T|^2\Theta_{T,k}]t^2\Big\} \tag{4.12}$$

The following proposition follows from the above construction via an argument of induction, using the $\kappa$-regular tame condition.

**Proposition 4.2.3.** *Each $\Theta_{T,j}$ can be written as a polynomial of $T$:*

$$\Theta_{T,j}(x,y) = \sum_{l=0}^{[\frac{j}{3}]+j} T^l\Theta_{l,j}(x,y),$$

*where $\Theta_{l,j}$ is independent of $T$, $[a]$ denotes the integral part of a real number $a$. Moreover*

$$|\Theta_{T,j}(x,y)| \leq C(\bar{V}_\gamma)^{\kappa' j}T^{[\frac{j}{3}]+j},$$

*where $\kappa' = \frac{\kappa+2}{3}$, $\bar{V}_\gamma = \sup_{p\in\gamma}|V(p)|$, $\gamma$ is the shortest geodesic connecting $x$ and $y$. When restricted to the diagonal of $M\times M$, $\Theta_{T,j}(y,y)$ can be written as an algebraic combination the curvature of the metric $g$, the function $f$, as well as their derivatives, at $y$; in addition, $\Theta_{T,0}(y,y) = \mathrm{Id}$.*

Let $\eta \in C_c^\infty(\mathbb{R})$ be a bump function, such that the support of $\eta$ is contained in $[-1,1]$, and $\eta|_{[-\frac{1}{2},\frac{1}{2}]} \equiv 1$. Let $\phi \in C^\infty(M\times M)$ be defined as

$$\phi(x,y) = \eta(d^2(x,y)/\tau^2). \tag{4.13}$$

**Proposition 4.2.4.** *Set*

$$K_{Tf}^k(t,x,y) = \phi(x,y)\mathcal{E}_0(t,x,y)\mathcal{E}_{1,T}(t,x,y)\sum_{j=0}^{k} t^j \Theta_{T,j}(x,y),$$

*then*

$$(\frac{\partial}{\partial t} + \Box_{Tf})K_{Tf}^k(t,x,y) = t^k\phi(x,y)R_{k,T}(t,x,y) + \Delta\phi(x,y)K_{Tf}^k(t,x,y) \\ - 2(\nabla\phi(x,y), \nabla K_{Tf}^k(t,x,y)),$$

*where $R_{k,T}$ is given by (4.12).*

The following lemma provides the estimate saying that $K_{Tf}^k(t,x,y)$ is a suitable parametrix for the heat kernel of the Witten Laplacian. The proof uses Lemma 4.3.9 which will be shown in the next section when we introduce the necessary notions.

**Lemma 4.2.5.** *Assume $t \in (0,1]$. Let*

$$\tilde{R}_{k,T} = t^k\phi(x,y)R_{k,T}(t,x,y) + \Delta\phi(x,y)K_{Tf}^k(t,x,y) - 2(\nabla\phi(x,y), \nabla K_{Tf}^k(t,x,y),$$

*then for $T \in (0, t^{-\frac{1}{2}}]$, any $a \in (0,1)$,*

$$|\tilde{R}_{k,T}(x,y)| \le C_{a,k}\chi_{B_x}(y)t^{(1-\kappa')k-\kappa'-\frac{n}{2}}T^{\frac{-2k+4}{3}}\exp(-atT^2h(x,y))\exp(-\frac{ad^2(x,y)}{4t}).$$

*Here $C_{a,k}$ is a constant depends on $a, k$, $\kappa' = \frac{\kappa+2}{3}$ (from Proposition 4.2.3), $B_x = \{y \in M : d(x,y) < \tau\}$, and $\chi_{B_x}(y)$ denotes the characteristic function of $B_x$.*

*Proof.* Since the support of $\Delta\phi(x,y)$ and $\nabla\phi(x,y)$ is a subset of $\{(x,y) \in M \times M :$

$\frac{d^2(x,y)}{\tau^2} \in (\frac{1}{2}, 1)\}$, by Proposition 4.2.3, Lemma 4.3.9 and the fact that $0 < a < 1$,

$$
\begin{aligned}
&|\Delta\phi(x,y)K^k_{Tf}(t,x,y) + (\nabla\phi(x,y), \nabla K^k_{Tf}(t,x,y))| \\
&\leq C_{k,a}\chi_{B_x} t^{-\frac{n}{2}} \exp(-\frac{(1-a)d^2(x,y)}{4t}) \exp(-atT^2h(x,y)) \exp(-\frac{ad^2(x,y)}{4t}) \\
&\leq C_{k,a,k}\chi_{B_x} t^{(1-\kappa')k-\kappa'-\frac{n}{2}} \exp(-atT^2h(x,y)) \exp(-\frac{ad^2(x,y)}{4t}).
\end{aligned}
$$

The last inequality follows form the fact that the function $t^l \exp(-t) \leq C_l$ for $t \in (0,\infty), l > 0$.

Similarly, by Proposition 4.2.3, Lemma 4.3.9 and the fact that $tT^2 \leq 1$, we have

$$
\begin{aligned}
|t^k\phi(x,y)R_{k,T}| &\leq C_k\chi_{B_x} \sum_{j=k}^{k+2} t^{j-\frac{n}{2}} T^{4(j+1)/3} \bar{V}_\gamma^{\kappa'(j+1)} \exp(-t\, h_T(x,y)) \exp(-\frac{d^2(x,y)}{4t}) \\
&\leq C'_k\chi_{B_x} t^{k-\frac{n}{2}} T^{\frac{4(k+1)}{3}} \bar{V}_\gamma^{\kappa'(k+1)} \exp(-tT^2h(x,y))) \exp(-\frac{d^2(x,y)}{4t}) \\
&\leq C_{a,k}\chi_{B_x} t^{(1-\kappa')k-\kappa'-\frac{n}{2}} T^{\frac{-2k+4}{3}} \exp(-atT^2h(x,y))) \exp(-\frac{ad^2(x,y)}{4t}).
\end{aligned}
$$

This finishes the proof. □

## 4.3 Li-Yau's Parabolic Distance and Heat Kernel Estimate

With the construction of the parametrix and the error estimate in the last section, we are now faced with the task of proving that it gives the desired asymptotic expansion of heat kernel. To this end, we need to estimate the convolutions of these terms, which seem quite daunting. Remarkably we found that a parabolic distance that appeared previously in Li-Yau's famous work [45] on the Harnack estimate of the heat kernel of Schrödinger operators greatly simplifies the task, both computationally and conceptually.

Our inspiration actually comes from the path integral formalism of quantum mechanics.

Another remarkable feature of Li-Yau's parabolic distance is its connection with the Agmon distance [32], [20], [25], which we will use to establish the needed lower bound for the parabolic distance. The resulting pointwise asymptotic expansion of the heat kernel will then be strong enough to pass to the trace of the heat kernel in the noncompact setting.

Let $K^k_{Tf}$ be the parametrix of $\partial_t + \Box_{Tf}$ constructed in Section 4.2, i.e.

$$K^k_{Tf}(t,x,y) = \phi(x,y)\mathcal{E}_0(t,x,y)\mathcal{E}_{1,T}(t,x,y)\sum_{j=0}^{k} t^j \Theta_{T,j}(x,y),$$

where $\phi$ is the cut-off function defined in (4.13).

We define convolution of $f(t,x,y), g(t,x,y) \in \Gamma(E)$ as

$$(f * g)(t,x,y) = \int_0^t \int_M (f(t-s,x,z), g(s,z,y))_z dvol(z) ds.$$

Let $K_{Tf}$ denote the heat kernel of $\Box_{Tf}$. By the Duhamel Principle, we have

**Lemma 4.3.1.** *The heat kernel $K_{Tf}$ is given by*

$$K_{Tf}(t,x,y) = K^k_{Tf}(t,x,y) + (K^k_{Tf} * \sum_{l=1}^{\infty}(-1)^l (\tilde{R}_{k,T})^{*l})(t,x,y).$$

*Here*

$$\tilde{R}^{*l}_{k,T} = \underbrace{\tilde{R}_{k,T} * \ldots * \tilde{R}_{k,T}}_{l\ times}.$$

Motivated by the path integral formalism of quantum mechanics, for any piecewise smooth curve $c : [0,t] \mapsto M$, s.t. $c(0) = x, c(t) = y$, we define

$$S_{t,x,y}(c) = \int_0^t \left( \frac{|c'(s)|^2}{4} + T^2 V(c(s)) \right) ds.$$

Let $C_{t,x,y} := \{\, c : [0,t] \mapsto M \mid c \text{ is piecewise smooth, } c(0) = x, c(t) = y \,\}$. Define Li-Yau's parabolic (meta-)distance as following

$$\tilde{d}_T(t,x,y) := \inf_{c \in C_{t,x,y}} S_{t,x,y}(c). \tag{4.14}$$

The following lemma summarizing its fundamental properties follows mostly from the definition.

**Lemma 4.3.2.** *$\tilde{d}_T(t,x,y)$ is a parabolic (meta-)distance; that is*

- *$\tilde{d}_T(t,x,y) \geq 0$;*
- *$\tilde{d}_T(t,x,y) = \tilde{d}_T(t,y,x)$;*
- *for $0 \leq s \leq t$, we have*

$$\tilde{d}_T(t-s,x,y) + \tilde{d}_T(s,y,z) \geq \tilde{d}_T(t,x,z). \tag{4.15}$$

*Moreover,*

$$\tilde{d}_T(t,x,y) \leq \frac{d^2(x,y)}{4t} + t\, h_T(x,y). \tag{4.16}$$

The last inequality follows from taking a minimal geodesic $\tilde{c}$ connecting $x$ and $y$ and noting that $S_{t,x,y}(\tilde{c}) = \frac{d^2(x,y)}{4t} + t\, h_T(x,y)$. The inequality (4.16) connects the parabolic distance to our parametrix.

Conceptually the most crucial property of the parabolic distance for the estimation of the convolutions of the error terms is the triangle inequality (4.15). We illustrate this by an example. If $V_T = 0$, then $\tilde{d}_T(t,x,y) = \frac{d^2(x,y)}{4t}$. In this case, the triangle inequality (4.15) reduces to the well known

$$\frac{d^2(x,y)}{t-s} + \frac{d^2(y,z)}{s} \geq \frac{d^2(x,z)}{t},$$

which plays a crucial role in the classical asymptotic expansion for the heat kernel.

The following lemma will be also needed in the heat kernel estimate involving the convolutions, and whose proof follows from a standard argument of volume comparison.

**Lemma 4.3.3.** *For $x \in M$ and $\delta < \tau$, denote $B_x = \{y \in M : d(x,y) < \delta\}$. Then there exists $A = A(F_0, \tau, \delta, n) > 0$, s.t.*

$$\int_{B_x} \exp(-\frac{d^2(x,z)}{t}) dz \leq A t^{\frac{n}{2}}.$$

*Recall that $F_0$ is the curvature bound, $\tau$ is the injectivity radius bound.*

With these preparations we now turn to the estimation of the convolution terms in the Duhamel Principle, Lemma 4.3.1. From now on, we fix an integer $k$ sufficiently large so that

$$\alpha(k,\kappa,n) = \frac{1}{3}(1-\kappa)k - \frac{\kappa+2}{3} - \frac{n}{2} + 1 > 0.$$

**Lemma 4.3.4.** *Assume that $t \in (0,1]$ and $T \in (0, t^{-\frac{1}{2}}]$. Then for any $a \in (0,1)$, there exist $C = C(k, a, \kappa, \tau, F_0) > 0$, such that, for all $l \in \mathbb{N}$,*

$$|K^k_{Tf} * \tilde{R}^{*l}_{k,T}|(t,x,y) \leq \frac{C^l t^{\alpha l} T^{\beta l}}{l!} \exp(-a\tilde{d}_T(t,x,y)),$$

*where $\alpha = \alpha(k,\kappa,n)$ as above, and $\beta = \beta(k) = \frac{-2k+4}{3}$.*

*Proof.* Let $B_x = \{y \in M : d(x,y) < \tau\}$, then by the volume comparison, we have

$$vol(B_x) \le C_\tau. \tag{4.17}$$

From Lemma 4.2.5 and (4.16), we have for $T \in (0, t^{-\frac{1}{2}}]$, any $a \in (0,1)$,

$$|\tilde{R}_{k,T}(x,y)| \le C_{a,k}\chi_{B_x}(y)t^{\alpha-1}T^{\beta}\exp(-a\tilde{d}_T(t,x,y)).$$

Therefore by (4.15),

$$\begin{aligned}
|\tilde{R}^{*l}_{k,T}| &= \Big| \int_0^t \int_0^{t_1} \dots \int_0^{t_{l-2}} \int_M \dots \int_M \tilde{R}_{k,T}(t-t_1,x,z_1)\tilde{R}_{k,T}(t_1-t_2,z_1,z_2) \\
&\quad \times \tilde{R}_{k,T}(t_2-t_3,z_2,z_3)\cdots\tilde{R}_{k,T}(t_{l-1},z_{l-1},y)dvol(z_{l-1})\cdots dvol(z_1)dt_{l-1}dt_{l-2}\cdots dt_1 \Big| \\
&= \Big| \int_0^t \int_0^{t_1} \dots \int_0^{t_{l-2}} \int_{B_x} \dots \int_{B_{z_{l-2}}} \tilde{R}_{k,T}(t-t_1,x,z_1)\tilde{R}_{k,T}(t_1-t_2,z_1,z_2) \\
&\quad \times \tilde{R}_{k,T}(t_2-t_3,z_2,z_3)\cdots\tilde{R}_{k,T}(t_{l-1},z_{l-1},y)dvol(z_{l-1})\cdots dvol(z_1)dt_{l-1}dt_{l-2}...dt_1 \Big| \\
&\le (C_\tau C_{a,k})^l T^{\beta l}\exp(-a\tilde{d}_T(t,x,y)) \int_0^t \int_0^{t_1} \dots \int_0^{t_{l-2}} (t-t_1)^{\alpha-1}\cdots t_{l-1}^{\alpha-1}dt_{l-1}\cdots dt_1 \\
&\le \frac{C^l t^{\alpha l-1}T^{\beta l}}{(l-1)!}\exp(-a\tilde{d}_T(t,x,y)).
\end{aligned}$$

On the other hand, by Proposition 4.2.3, $K_k(t,x,y) \le Ct^{-\frac{n}{2}}\exp(-a'\tilde{d}_T(t,x,y))$, where, for our purpose, $a' \in (0,1)$ is chosen to be $a' = \frac{1+a}{2} = a+b$, with $b = \frac{1-a}{2} > 0$. Hence

$$
\begin{aligned}
|K_{Tf}^k * \tilde{R}_{k,T}^{*l}|(t,x,y) &\leq \frac{C^{l+1}T^{\beta l}}{(l-1)!}\int_0^t \int_{B_x} (t-s)^{-\frac{n}{2}} s^{\alpha l-1} \\
&\times \exp(-a'\tilde{d}_T(t-s,x,z))\exp(-a\tilde{d}_T(s,z,y))dvol(z)ds \\
&\leq \frac{C^{l+1}T^{\beta l}}{(l-1)!}\exp(-a\tilde{d}_T(t,x,y)) \\
&\times \int_0^t s^{\alpha l-1}\int_{B_x}(t-s)^{-\frac{n}{2}}\exp(-\frac{bd^2(x,y)}{4(t-s)})dvol(z)ds \\
&\leq \frac{AC^{l+1}t^{\alpha l}T^{\beta l}}{(\alpha l)(l-1)!}\exp(-a\tilde{d}_T(t,x,y)).
\end{aligned}
$$

Here in the last inequality, we have made use of Lemma 4.3.3. □

We summarize our discussion so far.

**Theorem 4.3.1.** *The heat kernel $K_{Tf}$ has the following complete pointwise asymptotic expansion. For any $x, y \in M$ such that $d(x,y) \leq 1/2\tau$,*

$$K_{Tf}(t,x,y) \sim \frac{1}{(4\pi t)^{\frac{n}{2}}}\exp(-d^2(x,y)/4t)\exp(-t\,h_T(x,y))\sum_{j=0}^{\infty} t^j\Theta_{T,j}(x,y),$$

*as $t \to 0$. Each $\Theta_{T,j}$ is a polynomial of $T$:*

$$\Theta_{T,j}(x,y) = \sum_{l=0}^{[\frac{j}{3}]+j} T^l\Theta_{l,j}(x,y),$$

*and, when restricted to the diagonal of $M \times M$, $\Theta_{l,j}(y,y)$ can be written as an algebraic combination of the curvature of the metric $g$, the function $f$, as well as their derivatives, at $y$; in addition, $\Theta_{T,0}(y,y) = \mathrm{Id}$. Moreover, we have the following remainder estimate.*

*For any $k$ sufficiently large and any $a \in (0,1)$,*

$$\begin{aligned}\left| K_{Tf}(t,x,y) - \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp(-d^2(x,y)/4t)\exp(-t\, h_T(x,y)) \sum_{j=0}^{k} t^j \Theta_{T,j}(x,y) \right| \\ \leq C t^{\frac{1}{3}(1-\kappa)k - \frac{\kappa+2}{3} - \frac{n}{2} + 1} T^{\frac{-2k+4}{3}} \exp(-a\tilde{d}_T(t,x,y)),\end{aligned}$$

*for $t \in (0,1]$ and $T \in (0, t^{-\frac{1}{2}}]$.*

**Remark 4.3.5.** *Here the choice for $t \in (0,1]$ and $T \in (0, t^{-\frac{1}{2}}]$ is for simplicity and convenience. Our discussion works for $t \in (0, t_0]$ and $T \in (0, T_0 t^{-\frac{1}{2}}]$ but the estimates will depend on those choices as well.*

Without an effective lower bound on the parabolic distance $\tilde{d}_T(t,x,y)$ in our non-compact setting, the pointwise asymptotic expansion for the heat kernel of the Witten Laplacian will not be very useful beyond recovering the classical expansion. In particular, in passing from the pointwise asymptotic expansion to the asymptotic expansion of the (global) heat trace, we need remainder estimates which can compensate for the divergent volume integral. Here we explore the interesting connection of the parabolic distance to the Agmon distance and establish such an effective lower bound.

Recall that, in our setting, the Agmon metric (Cf. [32], [20], [25]) is $T^2|\nabla f|^2 g$. For any piecewise smooth curve $c$ in $M$, denote $L_{Tf}(c)$ the Agmon length of $c$, i.e., the length of $c$ with respect to Agmon metric $T^2|\nabla f|^2 g$.

First of all, we note

**Lemma 4.3.6.** *Let $c \in C_{t,x,y}$ be a piecewise smooth curve.Then,*

$$S_{t,x,y}(c) \geq L_{Tf}(c). \tag{4.18}$$

*Proof.* This follows from an elementary inequality as

$$S_{t,x,y}(c) = \int_0^t \frac{|c'(s)|^2}{4} + T^2 V(c(s))ds \geq \int_0^t T|\nabla f|(c(s))|c'(s)|ds = L_{Tf}(c).$$

Thus the parabolic distance is bounded from below by the Agmon distance (but we actually will be using the Agmon length later). $\square$

The following lemma says that the Agmon length can be bounded from below effectively if the potential function varies considerably along a curve.

**Lemma 4.3.7.** *Let $c \in C_{t,x,y}$ be a piecewise smooth curve. If*

$$\inf_{s\in[0,t]} V(c(s)) \leq \frac{1}{2} \sup_{s\in[0,t]} V(c(s)),$$

*then there exists constant $\bar{\beta} > 0$ depending only on the bounds in the tameness condition, such that*

$$L_{Tf}(c) \geq \bar{\beta} T \sup_{s\in[0,t]} |V|^{1-\kappa}(\gamma(s)).$$

*Proof.* Set $\bar{V}_c := \sup_{s\in[0,t]} V(c(s))$. Then we can find an interval $[a,b] \subset [0,t]$, s.t. $V(c(a)) = \frac{\bar{V}_c}{2}$, $V(c(b)) = \bar{V}_c$ (or vice versa, $V(c(b)) = \frac{\bar{V}_c}{2}$, $V(c(a)) = \bar{V}_c$). Moreover, for all $s \in [a,b]$, $V(c(s)) \geq \frac{\bar{V}_c}{2}$.

Now by the $\kappa$-regular tame condition,

$$\begin{aligned}
\frac{\bar{V}_c}{2} &= |V(c(a)) - V(c(b))| \leq \int_a^b |\nabla V(c(s))||c'(s)|ds \\
&\leq C \int_a^b |V(c(s))|^{\frac{\kappa+2}{2}} |c'(s)|ds \\
&\leq C \bar{V}_c^{\frac{\kappa+1}{2}} \int_a^b |\nabla f|(c(s))|c'(s)|ds \\
&\leq CT^{-1} \bar{V}_c^{\frac{\kappa+1}{2}} L_{Tf}(c|_[a,b])
\end{aligned}$$

Thus, for $\bar{\beta} = \frac{1}{2C} > 0$,

$$L_{Tf}(c) \geq L_{Tf}(c|_{[a,b]}) \geq \bar{\beta} T \bar{V}_c^{\frac{1-\kappa}{2}} = \bar{\beta} T \sup_{s\in[0,t]} |V|^{\frac{1-\kappa}{2}}(\gamma(s)). \tag{4.19}$$

□

Finally we arrive at the following effective lower bound for the parabolic distance.

**Lemma 4.3.8.** *One has*

$$\tilde{d}_T(t,x,y) \geq \min\{\bar{\beta} T V^{\frac{1-\kappa}{2}}(x), \frac{tT^2 V(x)}{2}\}. \tag{4.20}$$

*In particular, for $t \in (0,1], T = t^{-\frac{1}{2}}$,*

$$\tilde{d}_T(t,x,y) \geq \bar{\beta} V^{\frac{1-\kappa}{2}}(x) \min\{1, \frac{V(x)^{\frac{\kappa+1}{2}}}{2\bar{\beta}}\}. \tag{4.21}$$

*Proof.* Let $\gamma : [0,t] \mapsto M$ be a curve minimizing $S_{t,x,y}$. As before, set $\bar{V}_\gamma := \sup_{s\in[0,t]} V(\gamma(s))$.

If $V(\gamma(s)) \geq \frac{\bar{V}_\gamma}{2}$ for all $s \in [0,t]$, then we have

$$\tilde{d}_T(t,x,y) \geq \frac{tT^2\bar{V}_\gamma}{2} \geq \frac{tT^2 V(x)}{2}. \tag{4.22}$$

If not, by Lemma 4.3.7,

$$L_{Tf}(x,y) \geq \bar{\beta} T \bar{V}_\gamma^{\frac{1-\kappa}{2}} \geq \bar{\beta} T V^{\frac{1-\kappa}{2}}(x). \tag{4.23}$$

Therefore, by Lemma 4.3.6,

$$\tilde{d}_T(t,x,y) \geq \min\{\bar{\beta} T V^{\frac{1-\kappa}{2}}(x), \frac{tT^2 V(x)}{2}\}.$$

Our results follow. $\square$

We also note the following lemma which was used in the previous section.

**Lemma 4.3.9.** *For $a \in (0,1), t \in (0,1)$, $l > 0$, there exists $C_{a,\kappa,l} > 0$, s.t.*

$$\bar{V}_\gamma^l \exp(-\frac{d^2(x,y)}{4t}) \exp(-tT^2 h(x,y)) \le C_{a,\kappa,l} t^{-l} T^{-2l} \exp(-\frac{ad^2(x,y)}{4t}) \exp(-atT^2 h(x,y)),$$

*where $\gamma$ is the minimal geodesic connecting $x$ and $y$, $\bar{V}_\gamma = \sup_{p\in\gamma} |V(p)|$.*

*Proof.* When $\inf_{p\in\gamma} |V(p)| \ge \frac{\bar{V}_\gamma}{2}$, $h(x,y) \ge \frac{\bar{V}_\gamma}{2}$, hence $\bar{V}_\gamma^l \exp(-(1-a)tT^2 h(x,y)) \le C_{a,l} t^{-l} T^{-2l}$ for some $C_{a,l} > 0$.

Otherwise, by Lemmas 4.3.6 and 4.3.7, $\frac{d^2(x,y)}{4t} + tT^2 h(x,y) \ge \bar{\beta} T \bar{V}_\gamma^{\frac{1-\kappa}{2}} \ge \bar{\beta}(T^2 \bar{V}_\gamma)^{\frac{1-\kappa}{2}}$. Therefore, there exist $C_{a,\kappa,l}$ such that

$$\begin{aligned}\bar{V}_\gamma^l \exp(-(1-a)\frac{d^2(x,y)}{4t}) \exp(-(1-a)tT^2 h(x,y)) &\le \bar{V}_\gamma^l \exp(-(1-a)\bar{\beta}(T^2 V)^{1-\kappa}) \\ &\le C_{a,\kappa,l} T^{-2l} \le C_{a,\kappa,l} t^{-l} T^{-2l}\end{aligned}$$

which yields the result. $\square$

Combining the above discussion with Theorem 4.3.1 we have

**Theorem 4.3.2.** *For $T = t^{-\frac{1}{2}}$, the heat kernel $K_{t^{-\frac{1}{2}} f}$ of the Witten Laplacian has the following complete pointwise (diagonal) asymptotic expansion. For any $x \in M$,*

$$K_{t^{-\frac{1}{2}} f}(t,x,x) \sim \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp(-|\nabla f|^2(x)) \sum_{j=0}^{\infty} \sum_{l=0}^{[\frac{j}{3}]+j} t^{j-\frac{l}{2}} \Theta_{l,j}(x,x),$$

*as $t \to 0$. Moreover, for any $k$ sufficiently large and any $a \in (0,1)$,*

$$\left| K_{t^{-\frac{1}{2}}f}(t,x,x) - \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp(-|\nabla f|^2(x)) \sum_{j=0}^{\infty} \sum_{l=0}^{[\frac{j}{3}]+j} t^{j-\frac{l}{2}} \Theta_{l,j}(x,x) \right|$$
$$\leq C t^{\frac{1}{3}(2-\kappa)k - \frac{\kappa+1}{3} - \frac{n}{2}} \exp(-a\bar{\beta}|\nabla f|^{1-\kappa}(x)),$$

*for $t \in (0,1]$ and $x \in M$. In particular, we have the following small time asymptotic expansion of the heat trace:*

$$\mathrm{Tr}\left(\exp(-t\Box_{t^{-\frac{1}{2}}f})\right) \sim \frac{1}{(4\pi t)^{\frac{n}{2}}} \sum_{j=0}^{\infty} \sum_{l=0}^{[\frac{j}{3}]+j} t^{j-\frac{l}{2}} \int_M \exp(-|\nabla f|^2(x)) \,\mathrm{tr}(\Theta_{l,j}(x,x))dx,$$

*as $t \to 0$, with the remainder estimate*

$$\left| \mathrm{Tr}\left(\exp(-t\Box_{t^{-\frac{1}{2}}f})\right) - \frac{1}{(4\pi t)^{\frac{n}{2}}} \sum_{j=0}^{k} \sum_{l=0}^{[\frac{j}{3}]+j} t^{j-\frac{l}{2}} \int_M \exp(-|\nabla f|^2(x)) \,\mathrm{tr}(\Theta_{l,j}(x,x))dx \right|$$
$$\leq C t^{\frac{1}{3}(2-\kappa)k - \frac{\kappa+1}{3} - \frac{n}{2}}.$$

*Proof.* This follows from Theorem 4.3.1 and Lemma 4.3.8 by noting that $V \geq (2\bar{\beta})^{\frac{2}{\kappa+1}}$ outside a compact set. □

## 4.4 Local Index Theorem for Witten Laplacian

We now turn to the local index theorem for the Witten Laplacian. From the discussion at the end of Section 2 (see (4.6) and after) we have

$$\chi(M, d_{Tf}) = \sum_{i=0}^{n} (-1)^i dim(H^i_{(2)}(M, d_{Tf})) = \mathrm{Tr}_s(\exp(-t\Box_{Tf})) \tag{4.24}$$

is independent of $t$. Moreover, by Theorem 1.3 in [25], $\chi(M, d_{Tf})$ is independent of $T > 0$. As a consequence, Theorem 4.3.2 reduces the index formula for Witten Laplacian to a local index theorem, which we will develop in this section.

First we summarize what we know about the index of the Witten Laplacian as the following McKean-Singer type formula.

**Proposition 4.4.1.** *Assume that $(M, g, f)$ is polynomial tame. Then for $T > 0$, $\chi(M, d_{Tf})$ is independent of $T$ and*

$$\chi(M, d_{Tf}) = \int_M \mathrm{tr}_s(K_{Tf}(t, x, x))dx$$

*for any $t > 0$. Here $dx$ denotes the volume form induced by $g$.*

In the usual approach to the local index theorem, one studies the integrand, the pointwise supertrace $\mathrm{tr}_s(K_{Tf}(t, x, x))$, in the limit $t \to 0$ via the Getzler's rescaling. To proceed with Getzler's rescaling technique, we now fix $x_0 \in M$ and let $x$ be the normal coordinates near $x_0$. Thus $x = 0$ at $x_0$, and we will use 0 and $x_0$ interchangeably in this section. We trivialize the bundle $\Lambda^*(M)$ in the normal neighborhood $U$ by parallel transport along radical geodesic from $x_0$. In fact, we can assume $M = T_{x_0}M$ for now by extending everything trivially outside the normal neighborhood (we will see that we can localize the problem because of Theorem 4.3.2).

For usual Getzler's rescaling techniques (a la Bismut-Zhang [20] for the de Rham complex), one defines $\delta_\epsilon$ as follows:

1. For function $f \in C^\infty([0, \infty) \times U)$, $(\delta_\epsilon f)(t, x) = f(\epsilon t, \epsilon^{\frac{1}{2}} x)$. As a consequence, we have

$$\lim_{t \to 0} f(t, 0) = \lim_{\epsilon \to 0} (\delta_\epsilon f)(t, 0).$$

Moreover, $\delta_\epsilon f(t, x)\delta_\epsilon^{-1} = f(\epsilon t, \epsilon^{\frac{1}{2}} x)$, $\delta_\epsilon \partial_{x_i} \delta_\epsilon^{-1} = \epsilon^{-\frac{1}{2}} \partial_{x_i}$, $\delta_\epsilon \partial_t \delta_\epsilon^{-1} = \epsilon \partial_t$.

2. Let $\{e_i\}_{i=1}^n$ be a local frame near $x_0$, $\{e^i\}_{i=1}^n$ its dual frame. Then for $c(e_i) = e^i \wedge -\iota_{e_i}, \hat{c}(e_i) = e^i \wedge +\iota_{e_i}$, we define $\delta_\epsilon c(e_i) = \epsilon^{-\frac{1}{4}} e^i \wedge -\epsilon^{\frac{1}{4}} \iota_{e_i}$,$\delta_\epsilon \hat{c}(e_i) = \epsilon^{-\frac{1}{4}} e^i \wedge +\epsilon^{\frac{1}{4}} \iota_{e_i}$. Now let $c_\epsilon(e_i) = \epsilon^{-\frac{1}{4}} e^i \wedge -\epsilon^{\frac{1}{4}} \iota_{e_i}, \hat{c}_\epsilon(e_i) = \epsilon^{-\frac{1}{4}} e^i \wedge +\epsilon^{\frac{1}{4}} \iota_{e_i}$, then $\delta_\epsilon c(e_i) \delta_\epsilon^{-1} = c_\epsilon(e_i), \delta_\epsilon \hat{c}(e_i) \delta_\epsilon^{-1} = \hat{c}_\epsilon(e_i)$.

Recall that $K_{Tf}$ is the heat kernel of $\Box_{Tf}$. Then $K'_{Tf,\epsilon} = \epsilon^{\frac{n}{2}} \delta_\epsilon K_{Tf}$ is the heat kernel for $\Box'_{Tf,\epsilon} := \epsilon \delta_\epsilon \Box_{Tf} \delta_\epsilon^{-1}$. Moreover, for small $\epsilon$ [20, (4.60)]

$$\Box'_{Tf,\epsilon} = -\Delta_{T_{x_0}M} \operatorname{Id}_{\Lambda^* T^*_{x_0} M} + \frac{1}{2} \sum_{i<j<k<l} R_{ijkl}(0) e^i \wedge e^j \otimes \hat{e}^k \wedge \hat{e}^l + O(\epsilon^{\frac{1}{2}}),$$

where $\Delta_{T_{x_0}M}$ is the Euclidean Laplacian on $T_{x_0}M$, and $R_{ijkl}(x)$ is the Riemannian curvature tensor at $x$.

This is the usual Getzler's rescaling. As $\epsilon \to 0$, the information of $f$ disappears. But for the noncompact case, unlike the compact case, the index should depend on $f$. To deal with this issue, we introduce the following rescaling technique: we let $T$ join the game.

As mentioned before, the index $\chi(M, d_{Tf}) = \operatorname{Tr}_s(\exp(-t\Box_{Tf}))$ is independent of $T > 0$. Hence, in our rescaling, we define, in addition, $\delta_\epsilon(T) = \epsilon^{-\frac{1}{2}} T$.

Now under new rescaling, then we have

**Lemma 4.4.2.** *Let $\Box_{Tf,\epsilon} := \epsilon \delta_\epsilon \Box_{Tf} \delta_\epsilon^{-1}$. Then*

$$\Box_{Tf,0} := \lim_{\epsilon \to 0} \Box_{Tf,\epsilon} = -\Delta_{T_{x_0}M} \operatorname{Id}_{\Lambda^* T^*_{x_0} M} - \frac{1}{2} \sum_{i<j<k<l} R_{ijkl}(0) e^i \wedge e^j \otimes \hat{e}^k \wedge \hat{e}^l + V_T(x_0) + T L_{f,0}.$$

*Here $L_{f,0} = \nabla^2_{e_i,e_j} f(x_0) e_i \otimes \hat{e}_j$.*

*Proof.* By Proposition 4.1.1, $\Box_{Tf} = \Delta - TL_f + T^2|\nabla f|^2$. By [20, (4.60)],

$$\epsilon\delta_\epsilon \Delta \delta_\epsilon^{-1} = -\Delta_{T_{x_0}M}\,\mathrm{Id}_{\Lambda^* T^*_{x_0}M} + \frac{1}{2}\sum_{i<j<k<l} R_{ijkl}(0) e^i \wedge e^j \otimes \hat{e}^k \wedge \hat{e}^l + O(\epsilon^{\frac{1}{2}}).$$

On the other hand, by the new rescaling in $T$, $\epsilon\delta_\epsilon(T^2|\nabla f|^2)\delta_\epsilon^{-1} = T^2|\nabla f|^2(x_0) + O(\epsilon^{\frac{1}{2}})$. Now $L_f = \nabla^2_{e_i,e_j} f[e^i\wedge, \iota_{e_j}] = -\nabla^2_{e_i,e_j} f c(e^i)\hat{c}(e^j)$. Hence

$$\epsilon\delta_\epsilon(TL_f)\delta_\epsilon^{-1} = -T\nabla^2_{e_i,e_j} f(x_0) e_i \otimes \hat{e}_j + O(\epsilon^{\frac{1}{2}}).$$

Our result follows. $\square$

Denote $\tilde{R}(x_0) = -R_{ijkl}(x_0)e^i \wedge e^j \otimes \hat{e}^k \wedge \hat{e}^l$. Let $K_{Tf,0}$ be the heat kernel of $\Box_{Tf,0}$. Clearly $-\Delta_{T_{x_0}M}\,\mathrm{Id}_{\Lambda^* T^*_{x_0}M}$ commutes with $\frac{\tilde{R}(x_0)}{2} + TL_f(x_0) + V_T(x_0)$. Therefore we have

$$K_{Tf,0} = \mathcal{E}_0 \exp(-t[\frac{\tilde{R}(x_0)}{2} + TL_f(x_0) + V_T(x_0)]). \tag{4.25}$$

By Theorem 4.3.2, $K_{Tf}(t,x,x)$ has the following asymptotic expansion,

$$K_{Tf}(t,x,x) = (4\pi t)^{-\frac{n}{2}} \exp(-tV_T) \sum_{j=0}^{\infty} t^j \Theta_{T,j}(x,x),$$

with strong remainder estimate when $T = t^{-1/2}$. In particular,

$$K_{t^{-\frac{1}{2}}f}(t,x,x) = (4\pi t)^{-\frac{n}{2}} \exp(-V) \sum_{k\in\frac{1}{2}\mathbb{N}} t^k \sum_{j-\frac{1}{2}l=k,\, l\le j+[\frac{j}{3}]} \Theta_{l,j}(x,x). \tag{4.26}$$

Here $\mathbb{N}$ denotes the set of natural numbers which by our convention contains 0. Thus we can upgrade Proposition 4.4.1 to

**Proposition 4.4.3.** *For $T > 0$,*

$$\chi(M, d_{Tf}) = \lim_{t\to 0} \mathrm{Tr}(\exp(-t\Box_{t^{-\frac{1}{2}}f})) = \int_M \lim_{t\to 0} \mathrm{Tr}_s^{\Lambda^*(TM)}(K_{t^{-\frac{1}{2}}f}(t,x,x))dx \tag{4.27}$$
$$= \frac{1}{(4\pi)^{\frac{n}{2}}} \int_M \exp(-|\nabla f|^2) \sum_{j-\frac{1}{2}l=\frac{n}{2}} \mathrm{tr}_s^{\Lambda^*(TM)}(\Theta_{l,j}(x,x))dx.$$

*Here (to emphasize) we use $\mathrm{tr}_s^{\Lambda^*(TM)}$ to denote the pointwise supertrace on $\Lambda^*(TM)$ which was previously denoted by $\mathrm{tr}_s$.*

Now for $I = \{i_1, ..., i_k\} \subset \{1, 2, ..., n\}$, $(i_1 < ... < i_k)$, denote $c(e_I) = c(e_{i_1})...c(e_{i_k})$, $\hat{c}(e_I) = \hat{c}(e_{i_1})...\hat{c}(e_{i_k})$. Write $\Theta_{l,j} = \sum_{I,J\subset\{1,2,...,n\}} \Theta_{l,j,I,J} c(e_I)\hat{c}(e_J)$. The following Proposition on the key property of the supertrace is well known.

**Proposition 4.4.4.** *For $I, J \subset \{1, 2, ..., n\}$,*

$$\mathrm{tr}_s^{\Lambda^*(TM)}\left(c(e_I)\hat{c}(e_J)\right) = \begin{cases} (-1)^{\frac{n(n+1)}{2}} 2^n, & \textit{if } I = J = \{1,2,...,n\} \\ 0, & \textit{otherwise.} \end{cases}$$

Thus $\mathrm{tr}_s^{\Lambda^*(TM)}(\Theta_{l,j}) = (-1)^{\frac{n(n+1)}{2}} 2^n \Theta_{l,j,I_n,I_n}$, where $I_n = \{1, 2, ..., n\}$. We now recall the Berezin integral formalism. For any $\omega \in \Omega^*(TM)\hat{\otimes}\Omega^*(TM)$, $I \subset 1, 2, ..., n$, we can write $\omega$ as

$$\omega := \sum_I w_I \hat{e}^I.$$

Then the Berezin integral $\int^B : \Omega^*(TM)\hat{\otimes}\Omega^*(TM) \mapsto \Omega^*(TM)$ is defined as

$$\int^B \omega = \omega_{I_n}.$$

The following lemma is also well known in local index theory and the Getzler rescaling technique.

**Lemma 4.4.5.** *We have*

$$\lim_{t\to 0}\mathrm{tr}_s^{\Lambda^*(TM)}(K_{t^{-\frac{1}{2}}f})(t,x_0,x_0)dx = (-1)^{\frac{n(n+1)}{2}}2^n\int^B \lim_{\epsilon\to 0}\epsilon^{\frac{n}{2}}(\delta_\epsilon K_{t^{-\frac{1}{2}}f})(t,x_0,x), \tag{4.28}$$

*provided that the right hand limit exists.*

*Proof.* Write $K_{t^{-\frac{1}{2}}f}(t,x_0,x)=\sum_{I,J\subset\{1,2,\dots,n\}}a_{I,J}(t,x)c(e_I)\hat{c}(e_J)$. By Proposition 4.4.4,

$$\mathrm{tr}_s^{\Lambda^*(TM)}(K_{t^{-\frac{1}{2}}f}(t,x_0,x_0)) = (-1)^{\frac{n(n+1)}{2}}2^n a_{I_n,I_n}(t,x_0).$$

On the other hand,

$$(\epsilon^{\frac{n}{2}}\delta_\epsilon K_{t^{-\frac{1}{2}}f})(t,x_0,x) = \sum_{I,J\subset\{1,2,\dots,n\}} a_{I,J}(\epsilon t,\epsilon^{\frac{1}{2}}x)\epsilon^{\frac{n}{2}}c_\epsilon(e_I)\hat{c}_\epsilon(e_J).$$

Hence,

$$\int^B \lim_{\epsilon\to 0}\epsilon^{\frac{n}{2}}(\delta_\epsilon K_{t^{-\frac{1}{2}}f})(t,x_0,x) = \lim_{\epsilon\to 0}a_{I_n,I_n}(\epsilon t,\epsilon^{\frac{1}{2}}x)e^1\wedge\cdots\wedge e^n = \lim_{t\to 0}a_{I_n,I_n}(t,x_0)dx.$$

Our result follows. $\square$

For the right hand side of the previous lemma, we have the following proposition.

**Proposition 4.4.6.** *There exists $a\in(0,1)$ such that*

$$|\epsilon^{\frac{n}{2}}(\delta_\epsilon K_{t^{-\frac{1}{2}}f})(t,x,x) - K_{t^{-\frac{1}{2}}f,0}(t,x,x)| \le C\epsilon t^{2-\kappa-\frac{n}{2}}\exp(-aV^{1-\kappa}).$$

*Proof.* Let $K_0(t,x,y)=\phi(x,y)K_{Tf,0}(t,x,y)$. Then by the tameness condition, for some $a\in(0,1)$ we have

$$|(\Box_{Tf,\epsilon}-\Box_{Tf,0})K_0(t,x,y)| \le C\chi_{B_x}(y)\epsilon t^{-\frac{n+1-\kappa}{2}}T^{-2}\exp(-\frac{ad(x,y)}{4t})\exp(-atT^2V(x)).$$

By the Duhamel principle,

$$\epsilon^{\frac{n}{2}}K_{Tf,\epsilon} - K_0 = (\epsilon^{\frac{n}{2}}K_{Tf,\epsilon}) * ((\square_{Tf,\epsilon} - \square_{Tf,0})K_0(t,x,y)).$$

On the other hand, $\epsilon^{\frac{n}{2}}K_{Tf,\epsilon} = \epsilon^{\frac{n}{2}}(\delta_\epsilon K^k_{Tf} + \sum_{l=1}^\infty \delta_\epsilon(K^k_{Tf} * \tilde{R}^{*l}_{k,T})$, and it is straightforward to check that

$$|\epsilon^{\frac{n}{2}}\delta_\epsilon K_{Tf}(t,x,y)| \leq C\chi_{B_x}t^{-\frac{n}{2}}\exp(-\frac{atT^2}{2}V(x))\exp(-\frac{ad^2(x,y)}{4t}).$$

Proceeding as in the previous section we finish the proof of the Proposition. □

Finally, we arrive at our local index theorem for the Witten Laplacian. Recall that $\widetilde{R}, \widetilde{\nabla}^2 f \in \Omega^*(M)\hat{\otimes}\Omega^*(M)$ are defined as (we abuse the notatin here by omitting the wedge product signs)

$$\widetilde{R}(x) = R_{ijkl}(x)e^ie^j\hat{e}^k\hat{e}^l, \qquad \widetilde{\nabla}^2 f(x) = \nabla^2_{e_i,e_j}f(x)e^i\hat{e}^j.$$

**Theorem 4.4.1.** *For any $x_0 \in M$, we have*

$$\lim_{t\to 0}\mathrm{Tr}_s^{\Lambda^*(TM)}(K_{t^{-\frac{1}{2}}f})(t,x_0,x_0) = \frac{(-1)^{[\frac{n+1}{2}]}}{\pi^{\frac{n}{2}}}\exp(-|\nabla f(x_0)|^2)\int^B \exp(-\frac{\widetilde{R}(x_0)}{2} - \widetilde{\nabla}^2 f(x_0)).$$

*In particular, for $T > 0$,*

$$\chi(M, d_{Tf}) = \frac{(-1)^{[\frac{n+1}{2}]}}{\pi^{\frac{n}{2}}}\int_M \exp(-|\nabla f|^2)\int^B \exp(-\frac{\widetilde{R}}{2} - \widetilde{\nabla}^2 f).$$

*Proof.* By (4.28) and Proposition 4.4.6,

$$\begin{aligned}
\lim_{t\to 0}\mathrm{tr}_s^{\Lambda^*(TM)}(K_{t^{-\frac{1}{2}}f})(t,x_0,x_0)dx &= (-1)^{\frac{n(n+1)}{2}}2^n\int^B \lim_{\epsilon\to 0}(\epsilon^{\frac{n}{2}}(\delta_\epsilon K_{t^{-\frac{1}{2}}f})(t,x_0,x)) \\
&= (-1)^{\frac{n(n+1)}{2}}2^n\int^B K_{t^{-\frac{1}{2}}f,0} \\
&= \frac{(-1)^{\frac{n(n+1)}{2}}2^n}{(4\pi t)^{\frac{n}{2}}}\int^B \exp(-t\frac{\tilde{R}(x_0)}{2}-t^{\frac{1}{2}}L_f(x_0)-|\nabla f(x_0)|^2) \\
&= \frac{(-1)^{[\frac{n+1}{2}]}}{\pi^{\frac{n}{2}}}\exp(-|\nabla f(x_0)|^2)\int^B \exp(-\frac{\tilde{R}(x_0)}{2}-\tilde{\nabla}^2 f(x_0)).
\end{aligned}$$

The second result then follows from Proposition 4.4.3. □

## 4.5 Examples From Landau-Ginzburg Models

In this section we will disucss in somewhat detail how our results apply to some examples coming from Landau-Ginzburg models. Some of our discussions benefited from those of [33].

Consider a triple $(M, g, f)$, where $(M, g)$ is a Kähler manifold with bounded geometry, and $f : M \longrightarrow \mathbb{C}$ a holomorphic function. In this case, one considers the Witten deformation of the $\bar{\partial}$-operator

$$\bar{\partial}_f = \bar{\partial} + \partial f\wedge : \Omega^k(M,\mathbb{C}) \longrightarrow \Omega^{k+1}(M,\mathbb{C}).$$

The corresponding Witten Laplacian is then $\Box_{\bar{\partial},f} = \bar{\partial}_f^*\bar{\partial}_f + \bar{\partial}_f\bar{\partial}_f^*$.

On the other hand, one can also consider the underlying real manifold $M$ with the Riemannian metric given by $g$, together with the potential function given by $2\mathrm{Re}f =$

$f + \bar{f}$. It follows from the Kähler identity that

$$2\Box_{\bar{\partial},f} = \Box_{2\mathrm{Re}f}.$$

As a consequence, $\chi(M, \bar{\partial}_f) = \chi(M, d_{\mathrm{Re}f})$.

A large class of Landau-Ginzburg models consists of $(\mathbb{C}^n, g_0, f)$ where $g_0$ is the Euclidean metric and $f : \mathbb{C}^n \to \mathbb{C}$ a so-called nondegenerate quasi-homogeneous polynomial. Here $f \in \mathbb{C}[z_1, \cdots, z_n]$ is a quasi-homogeneous (also known as weighted homogeneous) polynomial if there are positive rational numbers $q_1, \cdots, q_n$, called the weights, such that

$$f(\lambda^{q_1} z_1, \cdots, \lambda^{q_n} z_n) = \lambda f(z_1, \cdots, z_n),$$

for all $\lambda \in \mathbb{C}^*$. $f$ is called nondegenerate if $f$ contains no monomials of the form $z_i z_j$ for $i \neq j$ and 0 is the only critical point of $f$ (equivalently, the hypersurface $f = 0$ in the weighted projective space is non-singular). By the classification result of [46] (see also [47, Theorem 3.7]), if $f$ is nondegenerate, then $q_i \leq \frac{1}{2}, \forall i$ (and these weights are unique).

If $f$ is a nondegenerate quasi-homogeneous polynomial, then $(\mathbb{C}^n, g_0, f)$ (or equivalently, the corresponding real model) is polynomial tame. To see this, one uses a result from [48]. Indeed, it is shown in [48, Theorem 5.8] that if $f$ is a nondegenerate quasi-homogeneous polynomial, then there exists a constant $C > 0$ depending only on $f$ such that for all $(u_1, \cdots, u_n) \in \mathbb{C}^n$, and each $i = 1, \cdots, n$,

$$|u_i| \leq C \left( \sum_{j=1}^{n} |\frac{\partial f}{\partial z_j}(u_1, \cdots, u_n)| + 1 \right)^{\gamma_i}, \tag{4.29}$$

where $\gamma_i = \frac{q_i}{\min_j (1 - q_j)}$.

As $|\nabla \mathrm{Re} f|^2 = \sum_j |\frac{\partial f}{\partial z_j}|^2$, one obtains using the above estimate and quasi-homogeneity that for $m \geq 1$,

$$|\nabla^m \mathrm{Re} f| \leq C(|\nabla \mathrm{Re} f| + 1)^{\frac{1-m\min_j q_j}{\min_j(1-q_j)}},$$

where the constant $C$ now also depends on $m$, $n$. Since $q_j \leq \frac{1}{2}$, the exponent here

$$\frac{1-m\min_j q_j}{\min_j(1-q_j)} \leq 2(1-m\min_j q_j).$$

Thus, if we let $\kappa = \max\{0, 1-4\min_j q_j\} < 1$, then the real model here $(\mathbb{R}^{2n}, g_0, \mathrm{Re} f)$ is $\kappa$-regular tame.

**Remark 4.5.1.** *It is also clear from the above discussion that when $m\min_j q_j \geq 1$, we can choose $\kappa = 0$, and therefore the real model $(\mathbb{R}^{2n}, g_0, \mathrm{Re} f)$ is effectively 0-regular tame.*

Also from the estimate (4.29) and $q_j \leq \frac{1}{2}$ one deduces that

$$|z|^2 \leq C(|\nabla \mathrm{Re} f|^2 + 1).$$

It follows that

$$\int_{|\nabla \mathrm{Re} f|^2 \leq \lambda\}} (\lambda - |\nabla \mathrm{Re} f|^2)^{2n/2} dvol \leq \lambda^n \operatorname{Vol}(B(0, \sqrt{C(\lambda+1)})) \leq C' \lambda^{2n}.$$

And thus $(\mathbb{R}^{2n}, g_0, \mathrm{Re} f)$ is polynomial tame. Therefore, Theorem 4.4.1 yields the following formula for the Milnor number of $f$, which is stated in [33] under additional restriction on the weights of $f$.

**Corollary 4.5.1.** *If $f \in \mathbb{C}[z_1, \cdots, z_n]$ is a nondegenerate quasi-homogeneous polynomial, then*

$$\chi(\mathbb{C}^n, \bar{\partial}_f) = \frac{(-1)^n}{\pi^n} \int_{\mathbb{C}^n} \exp(-|\partial f|^2) |det(-\partial^2 f)|^2 dvol.$$

*Proof.* Theorem 4.4.1 applied to the real model $(\mathbb{R}^{2n}, g_0, \mathrm{Re}f)$ gives us

$$\begin{aligned}\chi(\mathbb{C}^n, \bar{\partial}_f) = \chi(\mathbb{R}^{2n}, d_{\mathrm{Re}f}) &= \frac{(-1)^{[\frac{2n+1}{2}]}}{\pi^n}\int_{\mathbb{R}^{2n}} \exp(-|\nabla \mathrm{Re}f|^2)\int^B \exp(-\widetilde{\nabla}^2\mathrm{Re}f)\\ &= \frac{(-1)^n}{\pi^n}\int_{\mathbb{R}^{2n}} \exp(-|\nabla \mathrm{Re}f|^2)(-1)^n\det(-\nabla^2\mathrm{Re}f)dvol\\ &= \frac{(-1)^n}{\pi^n}\int_{\mathbb{C}^n} \exp(-|\partial f|^2)|det(-\partial^2 f)|^2 dvol.\end{aligned}$$

□

In the remaining part of the section we discuss the asymptotic expansion of the heat trace for the Witten Laplacian of the Landau-Ginzburg model $(\mathbb{C}^n, g_0, f)$, or equivalently, its real model $(\mathbb{R}^{2n}, g_0, \mathrm{Re}f)$, for $f$ a nondegenerate quasi-homogeneous polynomial, but without setting $T = t^{-\frac{1}{2}}$ as before.

By Theorem 4.3.1, we have a pointwise asymptotic expansion for the heat kernel with remainder estimate, which we will specialize here on the diagonal. For any $k$ sufficiently large and any $a \in (0,1)$, there exists $C > 0$ such that for $t \in (0,1]$ and $T \in (0, t^{-\frac{1}{2}}]$,

$$\begin{aligned}&\left|K_{Tf}(t,x,x) - \frac{1}{(4\pi t)^n}\exp(-tT^2V(x))\sum_{j=0}^{k} t^j\Theta_{T,j}(x,x)\right|\\ &\leq Ct^{\frac{1}{3}(1-\kappa)k-\frac{\kappa+2}{3}-n+1}T^{\frac{-2k+4}{3}}\exp(-a\tilde{d}_T(t,x,x)).\end{aligned}$$

Here

$$V = |\nabla \mathrm{Re}f|^2 = \sum_j |\frac{\partial f}{\partial z_j}|^2.$$

We will first see that the remainder estimate is strong enough for the global heat trace, namely it is convergent when integrated on $\mathbb{C}^n$. By Lemma 4.3.8

$$\tilde{d}_T(t,x,x) \geq \min\{\bar{\beta}TV^{\frac{1-\kappa}{2}}(x), \frac{tT^2V(x)}{2}\}.$$

On the other hand, by [33, Lemma 3.11(i)], which follows from the fact that $f$ is a nondegenerate quasi-homogeneous polynomial,

$$tV(z_1, \cdots, z_n) \geq V(t^{\delta q_1} z_1, \cdots, t^{\delta q_n} z_n), \qquad \delta = \frac{1}{2\min_j(1-q_j)} \leq 1.$$

Now set

$$\Omega_t = \left\{ V \leq (\frac{2\bar{\beta}}{tT})^{\frac{2}{1+\kappa}} \right\}, \qquad \Omega_t^c = \mathbb{C}^n - \Omega_t.$$

Then on $\Omega_t$,

$$\tilde{d}_T(t,z,z) \geq \frac{tT^2 V(z)}{2} \geq \frac{1}{2} T^2 V(t^{\delta q_1} z_1, \cdots, t^{\delta q_n} z_n).$$

Hence,

$$\int_{\Omega_t} e^{-a\tilde{d}_T(t,z,z)} dvol \leq \int_{\mathbb{C}^n} e^{-\frac{1}{2}aT^2 V(t^{\delta q_1} z_1, \cdots, t^{\delta q_n} z_n)} dvol = t^{-2\delta|q|} C(a,T), \quad |q| = \sum_j q_j.$$

On $\Omega_t^c$, $\tilde{d}_T \geq \bar{\beta} T V^{\frac{1-\kappa}{2}}$. Thus,

$$\int_{\Omega_t^c} e^{-a\tilde{d}_T(t,z,z)} dvol \leq \int_{\mathbb{C}^n} e^{-\bar{\beta} T V^{\frac{1-\kappa}{2}}} dvol = C_1(\bar{\beta}, T).$$

And we arrive at

$$\int_{\mathbb{C}^n} e^{-a\tilde{d}_T(t,z,z)} dvol \leq t^{-2\delta|q|}|C(a,T) + C_1(\bar{\beta}, T).$$

We now look at the terms in the asymptotic expansion given by Theorem 4.3.1. For a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$ with $\alpha_i$ nonnegative integer, we denote $\partial^\alpha f = \frac{\partial^{|\alpha|} f}{\partial^{\alpha_1} z_1 \cdots \partial^{\alpha_n} z_n}$, $|\alpha| = \alpha_1 + \cdots \alpha_n$. From the construction in Section 4.2, $\Theta_{T,j}(z,z)$ is a linear combination of $\partial^{\alpha^1} f \cdots \overline{\partial^{\alpha^l} f}$, with $l \leq j$ and (non-trivial) multi-indeces $\alpha^1, \cdots, \alpha^l$ satisfying $|\alpha^1| + \cdots |\alpha^l| \leq 2j$.

At this point we make the further assumption that $f$ is homogeneous; namely

$$q_1 = \cdots = q_n,$$

and we denote the common value by $q$. Differentiating the equation for quasi-homogeneity gives,

$$\lambda^{q|\alpha|}(\partial^\alpha f)(\lambda^{q_1} z_1, \cdots, \lambda^{q_n} z_n) = \lambda\, \partial^\alpha f(z_1, \cdots, z_n),$$

from which one deduces that

$$tV(z_1, \cdots, z_n) = V(t^{\delta q} z_1, \cdots, t^{\delta q} z_n).$$

Hence,

$$\int_{\mathbb{C}^n} e^{-tT^2 V} \partial^{\alpha^1} f \cdots \partial^{\alpha^l} f dvol = t^{\delta q \sum_{i=1}^l |\alpha^i| - \delta l - 2n\delta q} C_{\alpha^1, \cdots, \alpha^l}(f),$$

where $C_{\alpha^1, \cdots, \alpha^l}(f)$ is a constant depending on $f$ and $\alpha^1, \cdots, \alpha^l$.

We now summarize our discussion as the following result. For convenience we set $T = 1$ here. (Thus, for homogeneous $f$, we don't need to couple $tT^2 = 1$ to get a local index theorem.)

**Theorem 4.5.2.** *For the Landau-Ginzburg model $(\mathbb{C}^n, g_0, f)$ where $f$ is a nondegenerate homogeneous polynomial with weight $q$, we have the following small time asymptotic expansion of the heat trace for the Witten Laplacian:*

$$\mathrm{Tr}\,(\exp(-t\Box_f)) \sim \frac{1}{(4\pi t)^n} \sum_{j=0}^{\infty} \sum_{l \le j} \sum_{\alpha^1, \cdots, \alpha^l} t^{j + \delta q \sum_{i=1}^l |\alpha^i| - \delta l - 2n\delta q} C_{\alpha^1, \cdots, \alpha^l}(f),$$

*as $t \to 0$, where $|\alpha^1| + \cdots |\alpha^l| \le 2j$. Moreover, for $k$ sufficiently large, and $t \in (0,1]$,*

$$\left| \mathrm{Tr}\left(\exp(-t\Box_f)\right) - \frac{1}{(4\pi t)^n} \sum_{j=0}^{k} \sum_{l \le j} \sum_{\alpha^1, \cdots, \alpha^l} t^{j+\delta q \sum_{i=1}^{l} |\alpha^i| - \delta l - 2n\delta q} C_{\alpha^1,\cdots,\alpha^l}(f) \right| \le C t^{\frac{k+1}{3} - n - 2n\delta q}.$$

*Here $\delta = \frac{1}{2(1-q)}$.*

*Proof.* We note that $\kappa = 0$ in this case. The result follows from combining the above discussion. $\square$

**Remark 4.5.2.** *A similar but different expansion is in [33], and without the remainder estimate.*

# Chapter 5

# Analytic Torsion on Noncompact Manifolds

## 5.1 Introduction

In this section, we assume that our vector bundle $E \to M$ is bounded:

**Definition 5.1.1.** *A bounded vector bundle is a triple $(E, M, \nabla^E)$, where $E \to M$ is a complex vector bundle over Riemannian manifold $(M, g)$, and $\nabla^E$ is a connection on $E$ whose curvature tensor $R^E$ as well as all its $m$-th covariant derivative $\nabla^m R^E$ is uniformly bounded.*

Thus, if $(M, g)$ has bounded geometry, all its tensor bundles are bounded. As another example, a flat vector bundle $F \longrightarrow M$ is always bounded.

Now let $(F, \nabla^F)$ be a flat vector bundle on $M$ and $g^F$ a (fiberwise) metric on $F$. We denote by $\Omega^*(M, F)$ the space of smooth forms on $M$ with coefficients in $F$ and $d^F$ the exterior differential on $\Omega^*(M, F)$. Given a potential function $f$, the Witten deformation in this setting is simply

$$d^F_{Tf} = d^F + Tdf\wedge : \Omega^*(M,F) \longrightarrow \Omega^{*+1}(M,F), \tag{5.1}$$

wheere $T$ is the deformation parameter. A Riemannian metric $g$ on $M$ together with a metric $g^F$ on $F$ gives rise to an $L^2$ metric on $\Omega^*_0(M,F)$, those with compact supports. The formal adjoint of the Witten deformation with respect to this $L^2$ metric will be denoted by $d^{F*}_{Tf}$. If $(M,g,f)$ is polynomial tame, the corresponding Witten Laplacian

$$\Box^F_{Tf} = d^{F*}_{Tf}d^F_{Tf} + d^F_{Tf}d^{F*}_{Tf} \tag{5.2}$$

is essentially self-adjoint in $L^2\Lambda^*(M,F)$, the $L^2$-completion of $\Omega^*_0(M,F)$.

## 5.2 Redefining Ray-Singer Analytic Torsion

In this section we give an equivalent definition of the Ray-Singer analytic torsion which will be crucial for the generalization to Landau-Ginzburg models. The re-definition is motivated by our local index theorem in [49].

### 5.2.1 Previous Results for Witten Laplacian

In this subsection we review some of our previous results on the Hodge theory and local index theory for Witten deformations on noncompact manifolds [25, 49]. These will play important role in our current discussion. Note that in these work the flat vector bundle is set to be the trivial line bundle. However the results extend with only notational change.

Let $(M,g,f)$ be polynomial tame and $(F,\nabla^F)$ be a flat vector bundle on $M$ and $g^F$ a metric on $F$. For any $T > 0$, let $d^F_{Tf} := d^F + Tdf\wedge$ be the Witten deformation as defined in (5.1). As usual, the metrics $g, g^F$ induce an inner product $(\cdot,\cdot)_{L^2}$ on $\Omega^*_0(M,F)$:

$$(\phi,\psi)_{L^2} = \int_M (\phi,\psi)_{g,g^F} dvol, \phi,\psi \in \Omega^*_0(M,F).$$

Let $L^2\Lambda^*(M,F)$ be the completion of $\Omega^*_0(M,F)$ with respect to $\|\cdot\|_{L^2}$. Then $d^F_{Tf}$ is an unbounded operator on $L^2\Lambda^*(M,F)$ with domain $\Omega^*_0(M,F)$. Also, it has a formal adjoint operator $d^{F*}_{Tf}$, with $\mathrm{Dom}(d^{F*}_{Tf}) = \Omega^*_0(M,F)$, such that

$$(d^F_{Tf}\phi,\psi)_{L^2} = (\phi, d^{F*}_{Tf}\psi)_{L^2}, \phi,\psi \in \Omega^*_0(M,F).$$

Set $\Delta^F_{Tf} = (d^F_{Tf} + \delta^F_{Tf})^2$ and denote $\Delta^F$ the Hodge Laplacian associated with $d^F$. Let

$$L_{\nabla f} = d^F i_{\nabla f} + i_{\nabla f} d^F \tag{5.3}$$

be the Lie derivative acting on $\Omega^*(M,F)$.

Let $\nabla^{F*}$ be the adjoint connection of $\nabla^F$, i.e., for any $s,t \in \Gamma(E), X \in \Gamma(TM)$,

$$Xh^F(s,t) = h^F(\nabla^F_X s,t) + h^F(s,\nabla^{F*}_X t),$$

set $\nabla^e = \frac{1}{2}(\nabla^F + \nabla^{F*})$, $\omega(F,g^F) = \nabla^{F*} - \nabla^F$. One can see easily that $d^F_{Tf} = e^i \wedge \nabla^F_{e_i} + Tdf\wedge$, $d^{F*}_{Tf} = -\iota_{e_i} \wedge \nabla^{F*} + \iota_{T\nabla f}$, where $\{e_i\}$ is an orthonormal frame, $\{e^i\}$ its dual frame.

Let $D_{Tf} = d^F_{Tf} + d^{F*}_{Tf}$, $D^e_{Tf} = c(e^i)\nabla^e_{e_i} + T\hat{c}(\nabla f)$, where for $X \in \Gamma(TM)$, let $X^* \in \Gamma(T*M)$ be its dual, then $c(X) = X^* \wedge -\iota_X$, $\hat{c}(X) = X^* \wedge +\iota_X$. One can see easily that

$$D_{Tf} = D^e_{Tf} - \frac{1}{2}\widehat{c}\,(e_i)\,\omega\left(F,g^F\right)(e_i)\,.$$

The following basic formula is from [20].

**Proposition 5.2.1.** *The Witten Laplacian $\Delta^F_{Tf}$ has the following expression:*

$$\begin{aligned}\Delta^F_{Tf} = \Delta^e + \frac{S}{4} + \frac{1}{8}\left\langle e_k, R^{TM}\left(e_i, e_j\right) e_\ell\right\rangle \\ c\left(e_i\right) c\left(e_j\right) \widehat{c}\left(e_k\right) \widehat{c}\left(e_\ell\right) + \frac{1}{4}\left(\omega\left(F, g^F\right)\left(e_i\right)\right)^2 \\ - \frac{1}{8}\left(c\left(e_i\right) c\left(e_j\right) - \widehat{c}\left(e_i\right) \widehat{c}\left(e_j\right)\right)\left(\omega\left(F, g^F\right)\right)^2\left(e_i, e_j\right) \\ - \frac{1}{4} c\left(e_i\right) \widehat{c}\left(e_j\right)\left(\nabla^F_{e_i}\omega\left(F, g^F\right)\left(e_j\right) + \nabla^F_{e_j}\omega\left(F, g^F\right)\left(e_i\right)\right) \\ - T\omega(F, g^F)(\nabla f) + TL_f + T^2|\nabla f|^2\end{aligned}$$

*where $L_f = \nabla^2_{e_i,e_j} f[e^i\wedge, \iota_{e_j}]$, $\Delta^e = -\nabla^e_{e_i}\nabla^e_{e_i} + \nabla^e_{\nabla^{TM}_{e_i} e_i}$, $\{e_i\}$ is a local frame on $TM$ and $\{e^i\}$ is the dual frame on $T^*M$, $S$ is the scalar curvature.*

*Moreover, let*

$$\Delta^{F,0}_{Tf} = \Delta^e + \frac{S}{4} + \frac{1}{8}\sum_{1\le i,j,k,\ell\le n}\left\langle e_k, R^{TM}\left(e_i, e_j\right) e_\ell\right\rangle - T\omega(F, g^F)(\nabla f) + TL_f + T^2|\nabla f|^2,$$

$$\begin{aligned}\Delta^{F,1}_{Tf} = -\frac{1}{8}\sum_{1\le i,j\le n}\left(c\left(e_i\right) c\left(e_j\right) - \widehat{c}\left(e_i\right)\widehat{c}\left(e_j\right)\right)\left(\omega\left(F, g^F\right)\right)^2\left(e_i, e_j\right) \\ - \frac{1}{4}\sum_{1\le i,j\le n} c\left(e_i\right)\widehat{c}\left(e_j\right)\left(\nabla^F_{e_i}\omega\left(F, g^F\right)\left(e_j\right) + \nabla^F_{e_j}\omega\left(F, g^F\right)\left(e_i\right)\right).\end{aligned}$$

*Then one has $\Delta^F_{Tf} = \Delta^{F,0}_{Tf} + \Delta^{F,1}_{Tf}$, $*\Delta^{F,0}_{Tf} = \Delta^{F,0}_{-Tf}*$ and $*\Delta^{F,1}_{Tf} = -\Delta^{F,1}_{-Tf}*$, where $*$ is the Hodge star operator.*

We denote the Friedrichs extension of $\Delta^F_{Tf}$ by $\square^F_{Tf}$. Since $(M, g, f)$ is polynomial tame, $\Delta^F_{Tf}$ is essentially self-adjoint (and hence $\Box^F_{Tf}$ is the unique self-adjoint extension). In [25] or Chapter 3, we proved that when $(M, g, f)$ is tame,

$$L^2\Lambda^*(M, F) = \ker \Box^F_{Tf} \oplus \mathrm{Im}\bar{d}^F_{Tf} \oplus \mathrm{Im}\bar{d}^{F*}_{Tf}, \tag{5.4}$$

where $\bar{d}^F_{Tf}$ and $\bar{d}^{F*}_{Tf}$ are the graph extensions of $d^F_{Tf}$ and $d^{F*}_{Tf}$ respectively.

Setting $\Omega^*_{(2)}(M, F, Tf) := \mathrm{Dom}(\bar{d}^F_{Tf}) \cap \Omega^*(M, F)$, we have a chain complex

$$\cdots \xrightarrow{d^F_{Tf}} \Omega^*_{(2)}(M, F, Tf) \xrightarrow{d^F_{Tf}} \Omega^{*+1}_{(2)}(M, F, Tf) \xrightarrow{d^F_{Tf}} \cdots .$$

Let $H^*_{(2)}(M, F, d^F_{Tf})$ denote the cohomology of this complex. In [25] or Chapter 3, we have shown that $H^*_{(2)}(M, F, d^F_{Tf}) \cong \ker \Box^F_{Tf}$, provided $(M, g, f)$ is well tame and $T$ is large enough.

In [49] or Chapter 4, we develop a framework for the asymptotic expansion of the heat kernel for the Witten Laplacian in the noncompact case and proved a local index theorem. In general the situation is very complicated but it simplifies when we couple the deformation parameter $T$ and the time parameter $t$ by setting $tT^2 = 1$. Below we summarize the results in the form that we need here.

Let $(M, g, f)$ be $(\kappa, \alpha)$-polynomial tame, and $K_{Tf}(t, x, y)$ denote the heat kernel of the Witten Laplacian $\Box^F_{Tf}$.

**Theorem 5.2.1.** *For $T = t^{-\frac{1}{2}}$, the heat kernel $K_{Tf}$ has the following complete pointwise asymptotic expansion on the diagonal. For any $x \in M$,*

$$K_{Tf}(t, x, x) \sim \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp(-|\nabla f|^2(x)) \sum_{j=0}^{\infty} t^j \Theta_{T,j}(x),$$

*as $t \to 0$. Each $\Theta_{T,j}$ is a polynomial of $T$:*

$$\Theta_{T,j}(x) = \sum_{l=0}^{[\frac{j}{3}]+j} T^l \Theta_{l,j}(x),$$

*and $\Theta_{l,j}(x)$ can be written as an algebraic combination of the curvature of the metric $g$, the function $f$, and the endomorphism $\omega(F, g^F)$, as well as their derivatives, at $x$; in addition, $\Theta_{T,0}(x) = \mathrm{Id}$. Furthermore, for any $k$ sufficiently large, any $a \in (0,1)$,*

$$\left| K_{t^{-\frac{1}{2}}f}(t,x,x) - \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp(-|\nabla f|^2(x)) \sum_{j=0}^{\infty} \sum_{l=0}^{[\frac{j}{3}]+j} t^{j-\frac{l}{2}} \Theta_{l,j}(x,x) \right|$$
$$\leq C t^{\frac{1}{3}(2-\kappa)k - \frac{\kappa+1}{3} - \frac{n}{2}} \exp(-a\bar{\beta}|\nabla f|^{1-\kappa}(x)),$$

*for $t \in (0,1]$, where $\bar{\beta} > 0$ is a constant depending only on the bounds in the tameness condition. In particular, we have the following small time asymptotic expansion of the heat trace:*

$$\mathrm{Tr}\left(\exp(-t\Box_{t^{-\frac{1}{2}}f})\right) \sim \frac{1}{(4\pi t)^{\frac{n}{2}}} \sum_{j=0}^{\infty} \sum_{l=0}^{[\frac{j}{3}]+j} t^{j-\frac{l}{2}} \int_M \exp(-|\nabla f|^2(x)) \,\mathrm{tr}(\Theta_{l,j}(x,x)) dx,$$

*as $t \to 0$. Finally, assuming that $g^F$ is flat in the sense that $\nabla^F g^F = 0$, then the following local index theorem holds,*

$$\lim_{t\to 0} \mathrm{Tr}_s^{\Lambda^*(TM)}(K_{t^{-\frac{1}{2}}f})(t,x,x) = \frac{(-1)^{[\frac{n+1}{2}]}}{\pi^{\frac{n}{2}}} \exp(-|\nabla f(x)|^2) \int^B \exp(-\frac{\widetilde{R}(x)}{2} - \widetilde{\nabla}^2 f(x)).$$

Here $\int^B$ denotes the Berezin integral, to be recalled in a moment, and $\tilde{R}, \tilde{\nabla}^2 f \in$

$\Omega^*(TM)\hat{\otimes}\Omega^*(TM)$ are defined as

$$\tilde{R} = -\sum_{i<j,k<l} R_{ijkl} e^i e^j \hat{e}^k \hat{e}^l, \qquad \tilde{\nabla}^2 f = \nabla^2_{e_i,e_j} f e^i \hat{e}^j \tag{5.5}$$

for some orthonormal frame $\{e_i\}$ in $TM$ and its dual frame $\{e^i\}$ in $T^*M$.

As we see from the above result, the existence of the asymptotic expansion for the heat trace and local index theorem is quite complicated and in general requires the coupling of the deformation parameter $T$ and the heat parameter $t$. Correspondingly the original definition of the Ray-Singer analytic torsion requires modification to take this into account.

### 5.2.2 Estimation of Integrals

**Proposition 5.2.2.** *Let $(M, g, f)$ be $(\kappa, \alpha)$-polynomial tame, then for $t > 0$, $l \in \mathbb{Z}^+$, one has*

$$\int_M |f||\nabla f|^l \exp(-t|\nabla f|^2)\mathrm{dvol} < \infty, \tag{5.6}$$

$$\int_M |\nabla f|^l \exp(-t\rho)\mathrm{dvol} < \infty. \tag{5.7}$$

*Here $\rho(x) = \tilde{d}(x, K)$ for some compact set $K \subset M$, $\tilde{d}(x, x_0)$ is the Agmon distance between $x$ and $x_0$ with respect to Agmon metric $|\nabla f|^2 g$. In particular, Theorem 4.3.1 and (5.6) tell us that $\mathrm{Tr}_s\left(f\exp(-t\Box_{Tf})\right)$ is of trace class for all $t, T > 0$. As a result, follows from the same arguments in [], we know that*

$$\frac{dt}{2t}\,\mathrm{Tr_s}\left[N\exp\left(-tD_T^2\right)\right] - dT\,\mathrm{Tr_s}\left[f\exp\left(-tD_T^2\right)\right] \tag{5.8}$$

*is a closed 1-form in $\mathbb{R}^+ \times \mathbb{R}^+$.*

*Proof.* The estimation of (5.7) follows from the similar arguments in the proof of Proposition 2.4 in [49].

For $\lambda > 1$, let $K_\lambda := \{p \in M : |\nabla f|^2 < \lambda\}$ and $\tilde{K}_\lambda$ be the path-connected components of $K_\lambda$ containing $K_1$. then $|\tilde{K}_\lambda| < C_M \lambda^\alpha$ for some $C_M > 0$. By tameness condition, we also have

$$|Hessf| \leq c_M(|\nabla f|^2 + 1)^{\frac{\kappa+1}{2}} \tag{5.9}$$

for some $c_M > 0$.

Moreover, we claim that whenever $r \leq \frac{1}{2c_M(\lambda+2)^{\frac{\kappa+2}{2}}}$, we must have $B_r(p) \subset \tilde{K}_{\lambda+1}$ for all $p \in \tilde{K}_\lambda$, where $B_r(p) := \{q \in M : d(q,p) \leq r\}$, $d$ is the distance with respect to $g^{TM}$.

Otherewise, there exists $r' \leq \frac{1}{2c_M(\lambda+2)^{\frac{\kappa+2}{2}}}$, $p' \in \tilde{K}_\lambda$, such that $B_{r'}(p')$ touch $\partial\tilde{K}_{\lambda+1}$, then on $B_{r'}(p')$,

$$|\nabla f|^2 \leq \lambda + 1. \tag{5.10}$$

Assume $q' \in B_{r'}(p') \cap \partial\tilde{K}_{\lambda+1}$. As a result, by (5.9) and (5.10),

$$1 = |\nabla f|^2(q') - |\nabla f|^2(p') \leq r'|Hessf||\nabla f| \leq 1/2,$$

which is a contradiction.

Fix $\lambda > 1$. Let $\gamma$ be a curve connecting $p \in \tilde{K}_\lambda$ and $K_1$, then by the claim $N(\gamma) \subset \tilde{K}_{\lambda+1}$, where $N(\gamma) := \{q \in M : d(q,\gamma) < \frac{1}{2c_M(\lambda+2)^{\frac{\kappa+2}{2}}}\}$. Let $\delta$ be the length of $\gamma$ with respect to $g^{TM}$, then there exists $c''_M > 0$ such that

$$\frac{c''_M\delta}{(\lambda+2)^{\frac{\kappa+2}{2}}} \leq |N(\lambda)| \leq |\tilde{K}_{\lambda+1}| \leq C_M(\lambda+2)^\alpha. \tag{5.11}$$

Since $|f(p)| \leq \sup_{q\in K_1}|f(q)| + (\lambda+2)\delta$, by (5.11), there exists $A_M > 0$, such that $|f(p)| \leq A_M(1+\lambda)^{\frac{\kappa+2}{2}+\alpha}$ whenever $p \in \tilde{K}_\lambda$. Since $\cup_{\lambda>1}\tilde{K}_\lambda = M$, proceed as what we did in the proof of Proposition 2.4 in [49], we have the estimation (5.6).

□

### 5.2.3 An Equality in Compact Cases

In this subsection, we will assume that $(M, g)$ is a closed manifold. Let $(F, \nabla^F)$ be a flat vector bundle with a metric $g^F$. Let $f$ be a Morse function on $M$, $\Box^F_{Tf}$ be the Witten deformed Laplacian with respect to $d^F_{Tf}$. We denote by $\Box^F$ the Hodge Laplacian with respect to $d^F$. As usual $N$ is the number operator on $\Omega^*(M, F)$.

For $1 \leq i \leq n$, let $M^i$ be the number of $x \in \mathrm{Crit}(f)$ of index $i$. Set

$$\chi(F) = \sum_0^n (-1)^i \dim H^i(M, F, d^F_f),$$

$$\chi'(F) = \sum_0^n (-1)^i i \dim H^i(M, F, d^F_f).$$

Clearly,

$$\chi(F) = \mathrm{rk}(F) \sum_{x \in \mathrm{Crit}(f)} (-1)^{\mathrm{ind}(x)}.$$

Set

$$\widetilde{\chi}'(F) = \mathrm{rk}(F) \sum_{x \in \mathrm{Crit}(f)} (-1)^{\mathrm{ind}(x)} \mathrm{ind}(x) = \mathrm{rk}(F) \sum_{i=0}^n (-1)^i i M^i,$$

$$\mathrm{Tr}_{\mathrm{s}}^{\mathrm{Crit}(f)}[f] = \sum_{r \in \mathrm{Crit}(f)} (-1)^{\mathrm{ind}(x)} f(x).$$

For $s >> 0$, consider the following holomorphic functions:

$$\begin{aligned}
\psi_{1,f}(s) &= \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} (\mathrm{Tr}_s(N \exp(-t\Box_f^F)) - \chi'(F)) dt, \\
\psi_{2,f}(s) &= \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} (\mathrm{Tr}_s(N \exp(-t\Box^F_{t^{-\frac{1}{2}}f}) - \chi'(F)) dt \\
&\quad + \frac{2}{\Gamma(s)} \int_0^1 t^{s-\frac{3}{2}} \,\mathrm{Tr}_s(f \exp(-t\Box^F_{t^{-\frac{1}{2}}f})) dt \\
&= \psi_{2,1,f}(s) + \psi_{2,2,f}(s),
\end{aligned}$$

where $\chi'(M,F) = \sum_{i=0}^n (-1)^i i \, b_i(M,F),\ b_i(M,F) = dim(H^i(M,F,d_f^F))$.

By the asymptotic expansion of heat kernels, $\psi_1$ and $\psi_2$ extend to meromorphic functions on $\mathbb{C}$ which are both holomorphic at $s = 0$. We begin with the following observation. First of all, define as in [20]

$$B_T = \frac{\tilde{R}}{2} + T\tilde{\nabla}^2 f + T^2 |\nabla f|^2 \in \Omega^*(TM) \hat{\otimes} \Omega^*(TM), \tag{5.12}$$

where $\tilde{R}$ and $\tilde{\nabla}^2 f$ are defined in (5.5).

**Proposition 5.2.3.** *When $n$ is odd, we have*

$$\frac{d}{ds}\psi_{1,f}(s)|_{s=0} = \frac{d}{ds}\psi_{2,f}(s)|_{s=0} + \int_M \frac{\theta}{2}\left(F, g^F\right) \int^B \widehat{df} \exp\left(-B_{T^2}\right),$$

*When $n$ is even, then*

$$\frac{d}{ds}\psi_1|_{s=0} = \frac{d}{ds}\psi_{2,f}|_{s=0} + 2rk(F) \int_M f \int^B \exp(-B_0) + \int_M \frac{\theta}{2}\left(F, g^F\right) \int^B \widehat{df} \exp\left(-B_{T^2}\right).$$

*Proof.* By theorem 7.10 in [20],

$$\mathrm{Tr}_s(N\exp(-t\Box_f^F)) = \frac{n}{2}\chi(F) + \alpha_{-1}\frac{1}{\sqrt{t}} + O(\sqrt{t}), \tag{5.13}$$

where $a_{-1} = \mathrm{rk}(F)\int_M \int^B L\exp(-B_0)$ and $L = \frac{1}{2}\sum_{i=1}^n e^i\hat{e}^i \in \Omega^*(TM)\hat{\otimes}\Omega^*(TM)$.

By a similar argument but using the rescaling techniques in [49], we deduce that

$$\mathrm{Tr}_s\left(N\exp(-t\Box_{t^{-\frac{1}{2}}f}^F)\right) = \frac{n}{2}\chi(F) + b_{-1}\frac{1}{\sqrt{t}} + O(\sqrt{t}), \tag{5.14}$$

with $b_{-1} = \mathrm{rk}(F)\int_M \int^B L\exp(-B_1)$. Also,

$$\mathrm{Tr}_s\left(f\exp(-t\Box_{t^{-\frac{1}{2}}f}^F)\right) = c_{-1} + \sqrt{t}d_{1/2} + O(t), \tag{5.15}$$

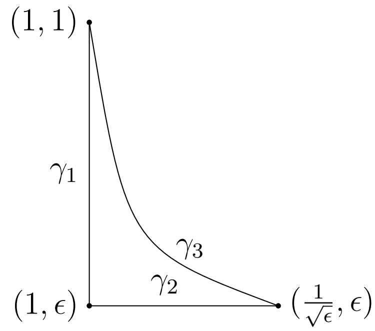
with $c_{-1} = \int_M \int^B f\exp(-B_1), d_{1/2} = \int_M \frac{\theta}{2}\left(F, g^F\right)\int^B \widehat{df}\exp\left(-B_{T^2}\right)$.

By Theorem 5.6 in [20], we know that

$$\frac{1}{2t}\frac{\partial}{\partial T}\mathrm{Tr}_s[N\exp(-t\Box_{Tf}^F)] = -\frac{\partial}{\partial t}\mathrm{Tr}_s[f\exp(-t\Box_{Tf}^F). \tag{5.16}$$

which means that

$$\alpha_{t,T} = \frac{dt}{2t}\mathrm{Tr}_s[N\exp(-t\Box_{Tf}^F)] - dT\,\mathrm{Tr}_s[f\exp(-t\Box_{Tf}^F) \tag{5.17}$$

is a closed 1-form in $\mathbb{R}_+ \times \mathbb{R}_+$, Hence for $\epsilon > 0$, we can see that the integral of $\alpha_{t,T}$ on the loop indicated here is zero: 

where $\gamma_3$ denotes the curve $T = t^{-\frac{1}{2}}$. We now examine the integral over each individual piece of the loop.

- Integral over $\gamma_1$:

$$\begin{aligned} I_{1,\epsilon} &:= \int_\epsilon^1 \mathrm{Tr}_s \left( N \exp(-t\Box_f^F) \right) \frac{dt}{2t} \\ &= \int_\epsilon^1 \left( \mathrm{Tr}_s \left( N \exp(-t\Box^F_{t^{-\frac{1}{2}}f}) \right) - a_{-1}\frac{1}{\sqrt{t}} - \frac{n}{2}\chi(F) \right) \frac{dt}{2t} \\ &- a_{-1}(1-\epsilon^{-\frac{1}{2}}) - \frac{n}{4}\chi(F)\log(\epsilon). \end{aligned} \tag{5.18}$$

  Hence, as $\epsilon \to 0$,

$$I_{1,\epsilon} - a_{-1}\epsilon^{-\frac{1}{2}} + (\frac{n}{4}\chi(F) + \chi'(F))\log(\epsilon) \to \frac{1}{2}\frac{d}{ds}\psi_1|_{s=0} \tag{5.19}$$

- Integral over $\gamma_2$:

By Theorem 7.12 in [20],

$$
\begin{aligned}
I_{2,\epsilon} &:= -\int_1^{\frac{1}{\sqrt{\epsilon}}} \mathrm{Tr}_s \left( f \exp(-\epsilon \Box^F_{Tf}) \right) dT \\
&= -\frac{1}{\sqrt{\epsilon}} \int_{\sqrt{\epsilon}}^1 rk(F) \int_M f \int^B \exp(-B_T) dT + \int_M \frac{\theta}{2} \left(F, g^F\right) \int^B \widehat{df} \exp\left(-B_{T^2}\right) + O(\epsilon) \\
&= -\frac{1}{\sqrt{\epsilon}} \int_0^1 rk(F) \int_M f \int^B \exp(-B_T) dT + rk(F) \int_M f \int^B \exp(-B_0) \\
&+ \int_M \frac{\theta}{2} \left(F, g^F\right) \int^B \widehat{df} \exp\left(-B_{T^2}\right) + O(\epsilon)
\end{aligned}
\tag{5.20}
$$

- Integral over $\gamma_3$:

$$
\begin{aligned}
I_{3,1,\epsilon} &:= -\int_\epsilon^1 \mathrm{Tr}_s \left( N \exp(-t \Box^F_{t^{-\frac{1}{2}} f}) \right) \frac{dt}{2t} \\
&= -\int_\epsilon^1 \left( \mathrm{Tr}_s \left( N \exp(-t \Box^F_{t^{-\frac{1}{2}} f}) \right) - b_{-1} \frac{1}{\sqrt{t}} - \frac{n}{2} \chi(F) \right) \frac{dt}{2t} \\
&+ b_{-1}(1 - \epsilon^{-\frac{1}{2}}) + (\frac{n}{4} \chi(F) + \chi'(F)) \log(\epsilon).
\end{aligned}
\tag{5.21}
$$

Hence, as $\epsilon \to 0$,

$$
I_{3,1,\epsilon} + b_{-1} \epsilon^{-\frac{1}{2}} - \frac{n}{4} \chi(F) \log(\epsilon) \to -\frac{1}{2} \frac{d}{ds} \psi_{2,1}|_{s=0}
\tag{5.22}
$$

$$
\begin{aligned}
I_{3,2,\epsilon} &:= \frac{1}{2} \int_\epsilon^1 t^{-3/2} \,\mathrm{Tr}_s \left( f \exp(-t \Box^F_{t^{-\frac{1}{2}} f}) \right) dt \\
&= \frac{1}{2} \int_\epsilon^1 t^{-3/2} \left( \mathrm{Tr}_s \left( f \exp(-t \Box^F_{t^{-\frac{1}{2}} f}) \right) - c_{-1} \right) dt \\
&- c_{-1}(1 - \epsilon^{-\frac{1}{2}})
\end{aligned}
\tag{5.23}
$$

Hence, as $\epsilon \to 0$,

$$I_{3,2,\epsilon} - c_{-1}\epsilon^{-\frac{1}{2}} \to \frac{1}{2}\frac{d}{ds}\psi_{2,2}|_{s=0} \tag{5.24}$$

Now looking at the divergent terms:

- For $\log(\epsilon)$ term:

  By (5.18) and (5.21), the coefficient of $\log(\epsilon)$ vanished.

- For $\frac{1}{\sqrt{\epsilon}}$ term, by (5.18), (5.20), (5.21) and (5.23), the coefficient is

  $$b_{-1} - a_{-1} - c_{-1} - \int_0^1 \int_M f \exp(-B_T) dT,$$

  which is zero by Theorem 3.17 in [20].

As a consequence, by (5.19), (5.22) and (5.24) and the fact that $I_{1,\epsilon}+I_{2,\epsilon}+I_{3,1,\epsilon}+I_{3,2,\epsilon} = 0$

$$\frac{d}{ds}\psi_1|_{s=0} = \frac{d}{ds}\psi_{2,f}|_{s=0} + 2rk(F)\int_M f\int^B \exp(-B_0) + \int_M \frac{\theta}{2}\left(F, g^F\right)\int^B \widehat{df}\exp\left(-B_{T^2}\right).$$

Notice that when $n$ is odd, $\int^B \exp(-B_0) = 0$, which finishes the proof.

□

### 5.2.4 Redefining Ray-Singer Metric

Now we can redefine our Ray-Singer metric $\|\cdot\|^{RS}_{H^*(M,F,d_f)}$ to be

**Definition 5.2.4.**

$$\log\|\cdot\|^{RS}_{H^*(M,F,d_f)} := \log|\cdot|^{RS}_{H^*(M,F,d_f)}\frac{d}{ds}\theta(s)|_{s=0},$$

*where*

$$\theta(s) = \frac{1}{\Gamma(s)} \int_1^\infty t^{s-1} \left(\mathrm{Tr}_s \left(N \exp(-t\Box_f^F)\right) - \chi'(M,F)\right) dt + \psi_2(s).$$

**Proposition 5.2.5.** *If $f_\tau$ is a family of Strongly polynomial tamed function, then when $n$ is odd,*

$$\log \|\cdot\|^{RS}_{H^*(M,F,d_{f_\tau})}$$

*is independent of $\tau$.*

*Proof.* Let $\phi = \partial_\tau f_\tau|_{\tau=0}$. By (5.16), one can see that

$$\begin{aligned}
&\partial_\tau \int_1^\infty t^{s-1} \left(\mathrm{Tr}_s \left(N \exp(-t\Box_f^F)\right) - \chi'(M,F)\right) dt|_{\tau=0} = 2\,\mathrm{Tr}_s(\phi \exp(-\Box_f)) - 2\,\mathrm{Tr}_s(\phi P) \\
&- 2s \int_1^\infty t^{s-1} \left(\mathrm{Tr}(\phi \exp(-t\Box_f))\right) dt.
\end{aligned} \tag{5.25}$$

Moreover, by (5.16) again, if $s >> 0$,

$$\begin{aligned}
&\partial_\tau \int_0^1 t^{s-1} (\mathrm{Tr}_s(N \exp(-t\Box^F_{t^{-\frac{1}{2}}f}) - \chi'(M,F))dt \\
&= \int_0^1 -2t^{s-1/2} \partial_t (\mathrm{Tr}_s(\phi \exp(-t\Box^F_{t^{-\frac{1}{2}}f}))dt + 1/2 \int_0^1 -2t^{s-2} \partial_T (\mathrm{Tr}_s(\phi \exp(-t\Box^F_{Tf}))dt\Big|_{T=t^{-\frac{1}{2}}} \\
&= (s-1/2) \int_0^1 2t^{s-3/2}\, \mathrm{Tr}_s(\phi \exp(-t\Box^F_{t^{-\frac{1}{2}}f})dt + 1/2 \int_0^1 -2t^{s-2} \partial_T (\mathrm{Tr}_s(\phi \exp(-t\Box^F_{Tf}))dt\Big|_{T=t^{-\frac{1}{2}}} \\
&- 2\,\mathrm{Tr}_s(\phi \exp(-\Box_f)).
\end{aligned} \tag{5.26}$$

$$\begin{aligned}
&\partial_\tau \int_0^1 t^{s-\frac{3}{2}}\, \mathrm{Tr}_s(f \exp(-t\Box^F_{t^{-\frac{1}{2}}f}))dt \\
&= \int_0^1 t^{s-3/2}\, \mathrm{Tr}_s(\phi \exp(-t\Box^F_{t^{-\frac{1}{2}}f})dt + \int_0^1 t^{s-3/2} (\mathrm{Tr}_s(f \partial_\tau \exp(-t\Box^F_{t^{-\frac{1}{2}}f}))dt
\end{aligned} \tag{5.27}$$

One can see that

$$2t^{s-3/2}(\mathrm{Tr}_s(f\partial_\tau \exp(-t\Box^F_{t^{-\frac{1}{2}}f})) = t^{s-2}\partial_T(\mathrm{Tr}_s(\phi\exp(-t\Box^F_{Tf}))|_{T=t^{-\frac{1}{2}}} \tag{5.28}$$

By (5.25) (5.26), (5.27) and ((5.28)), one has

$$\begin{aligned}&\partial_\tau\int_1^\infty t^{s-1}\left(\mathrm{Tr}_s\left(N\exp(-t\Box^F_f)\right)-\chi'(M,F)\right)dt+\partial_\tau\psi_2\\&=-2\,\mathrm{Tr}_s(\phi P)-2s\int_1^\infty t^{s-1}\left(\mathrm{Tr}(\phi\exp(-t\Box_f))\right)dt\\&+s\int_0^1 2t^{s-3/2}\,\mathrm{Tr}_s(\phi\exp(-t\Box^F_{t^{-\frac{1}{2}}f})dt.\end{aligned} \tag{5.29}$$

As a result

$$\partial_\tau\log\|\cdot\|^{RS}_{H^*(M,F,d_f)}=0.$$

$\square$

By Proposition 5.2.3, on closed manifold, we have

**Proposition 5.2.6.**

$$2\log\|\cdot\|^{RS}_{H^*(M,F,d)}=\log\|\cdot\|^{RS}_{H^*(M,F,d_f)}+\log\|\cdot\|^{RS}_{H^*(M,F,d_{-f})}$$

Let $R(M,g,h,F,f)=\log\|\cdot\|^{RS}_{H^*(M,F,d_f)}+\log\|\cdot\|^{RS}_{H^*(M,F,d_{-f})}$, where $n$ is the dimension of the manifolds $M$. Next, step, we would like to study the variation of $R$, that is, suppose we have a family of metric $g_l$ on $M$ and $h_l$ on $F$, we would like to compute $\frac{\partial}{\partial l}R(M,g_l,h_l,F,f)$. To this end, using the same notation as in [20], let $L:=*_l^{-1}\frac{\partial}{\partial l}*_l+h_l^{-1}\frac{\partial}{\partial l}h_l$

We may also need Proposition 2.4 and Proposition 2.5 in [50], for the self-containess, we restate them as below:

**Proposition 5.2.7.** *A Riemannian $n$ -manifold $M$ has bounded geometry if and only if there is a ball $B$ with center 0 in $\mathbb{R}^n$ such that*

- *$B$ is the domain of a normal coordinate system at every point* m *of $M$*
- *the Christoffel symbols of $M$, considered as a family of smooth functions parametrized by indices $i, j$, and $k$ and by a point m of $M$, lie in a bounded subset of the Fréchet space $C^\infty(B)$.*

*Such a ball is called a good coordinate ball.*

**Proposition 5.2.8.** *Let $M$ be a manifold of bounded geometry. A Clifford bundle $S$ over $M$ has bounded geometry if and only if there is a good coordinate ball $B$ such that the Christoffel symbols for $S$ lie in a bounded subset of $C^\infty(B)$.*

Given a family of metric $(g_l, h_l)$, $l \in [0,1]$ on $TM \to M$ and $F \to M$, let $\nabla_l^{TM}$ and $\nabla_l f$ be the Levi-Civita connection and gradient induced by $g_l$, $\tilde{R}_l$ and $(\tilde{\nabla}^2)_l f$ are operators induced by $g_l$.

**Proposition 5.2.9.** *Suppose the variation of metric is controllable, then when $n$ is even one has anomaly formal:*

$$\begin{aligned}\frac{\partial}{\partial l} R(M, g_l, h_l, F, f) = 2 \int_M \operatorname{tr}\left[\left(h_l^F\right)^{-1} \frac{\partial h_l^F}{\partial \ell}\right] \frac{(-1)^{\left[\frac{n+1}{2}\right]}}{\pi^{\frac{n}{2}}} \int^B \exp\left(-\frac{\widetilde{R}_l(x) + |\nabla_l f|^2 - (\tilde{\nabla}^2)_l f(x)}{2}\right) \\ - 2\int_M \theta\left(F, g'^F\right)_l \tilde{e}_l\left(TM, \nabla^{TM}, \nabla_l^{TM}, f\right),\end{aligned}$$

*where $\theta_l(F, h^F) = \operatorname{Tr}((h_l)^{-1} h_l)$, $\tilde{e}_l(TM, \nabla^{TM}, \nabla_l^{TM}, f)$ is the Chern-Simon transgressed form;*

$$\tilde{e}_l(TM, \nabla^{TM}, \nabla_l^{TM}, f) = \int_0^l \int^B \left(\frac{\partial}{\partial l}\nabla_l^{TM} + \widehat{\frac{\partial}{\partial l}(\nabla_l f)^{\#}}\right) \exp\left(-\frac{\widetilde{R}_l(x) + |\nabla_l f|^2 - (\tilde{\nabla}^2)_l f(x)}{2}\right) dl.$$

*When $n$ is odd,* $\frac{\partial}{\partial l}R(M, g_l, h_l, F, f) = 0.$

*Proof.* For simplicity, let's assume that $H^*(M, F) = 0$ first. Let $g_l$, $h_l$ be a family of metric on $M$ and $F$ correspondingly.

First, we compute $\frac{\partial}{\partial l}\left(\psi_{2,1,f}(s) + \psi_{2,1,-f}(s)\right)$: We notice that $\frac{\partial}{\partial l}d_f^{F,*} = -[L, d_f^{F,*}]$. Proceed as in the proof of Theorem 5.6 in [20], one can show that

$$\frac{\partial}{\partial l}Tr_s[N\exp(-t\Box_{Tf})] = -t\frac{\partial}{\partial t}Tr_s[L\exp(-t\Box_{Tf})] \tag{5.30}$$

By (5.30), we can see that

$$Tr_s[L\exp(-t\Box_{Tf})] = -\int_\epsilon^t \frac{1}{s}\frac{\partial}{\partial l}Tr_s[N\exp(-s\Box_{Tf})]ds + Tr_s[L\exp(-\epsilon\Box_{Tf})] \tag{5.31}$$

By (5.31) and Theorem 5.6 in [20], one has

$$\begin{aligned}
\frac{\partial}{\partial T}Tr_s[L\exp(-t\Box_{Tf})] &= -\int_\epsilon^t \frac{1}{s}\frac{\partial}{\partial T}\frac{\partial}{\partial l}Tr_s[N\exp(-s\Box_{Tf})]ds + \frac{\partial}{\partial T}Tr_s[L\exp(-\epsilon\Box_{Tf})] \\
&= \int_\epsilon^t 2\frac{\partial}{\partial l}\frac{\partial}{\partial s}Tr_s[f\exp(-s\Box_{Tf})]ds + \frac{\partial}{\partial T}Tr_s[L\exp(-\epsilon\Box_{Tf})] \\
&= 2\frac{\partial}{\partial l}Tr_s[f\exp(-t\Box_{Tf})] - 2\frac{\partial}{\partial l}Tr_s[f\exp(-\epsilon\Box_{Tf})] + \frac{\partial}{\partial T}Tr_s[L\exp(-\epsilon\Box_{Tf})]
\end{aligned}$$

Now we would like to compute

$$\lim_{\epsilon\to 0} -2\frac{\partial}{\partial l}Tr_s[f\exp(-\epsilon\Box_{Tf})] + \frac{\partial}{\partial T}Tr_s[L\exp(-\epsilon\Box_{Tf})] + n.p.$$

Here $n.p.$ denotes the terms that we replace $f$ in previous terms by $-f$.

To this end, we compute

$$
\begin{aligned}
&2\frac{\partial}{\partial l}\operatorname{Tr}_s\left(f\exp(-\epsilon\Box_{Tf})\right)+n.p. = -2\epsilon\frac{\partial}{\partial b}\operatorname{Tr}_s\left(f\exp(-\epsilon\Box_{Tf}-b[D_{Tf},\frac{\partial}{\partial l}D_{Tf}])\right)|_{b=0}+n.p.\\
&=-2\epsilon\operatorname{Tr}_s\left([D_{Tf},f]\exp(-\epsilon\Box_{Tf}-b\frac{\partial}{\partial l}D_{Tf})\right)|_{b=0}+n.p.\\
&=-2\epsilon\operatorname{Tr}_s\left(c(df)\exp(-\epsilon\Box_{Tf}-b[(d_T^{F*}),L])\right)|_{b=0}+n.p.\\
&=-2\epsilon\operatorname{Tr}_s\left([d_T^{F*},c(df)]\exp(-\epsilon\Box_{Tf}-bL)\right)|_{b=0}+n.p.\\
&=-\epsilon\operatorname{Tr}_s\left((2T|\nabla f|^2-\hat{c}(e_i)c(\nabla_{e_i}\nabla f)+c(e_i)c(\nabla_{e_i}\nabla f)-2\nabla_{\nabla f})\exp(-\epsilon\Box_{Tf}-bL)\right)|_{b=0}+n.p.\\
&=-\epsilon\operatorname{Tr}_s\left((2T|\nabla f|^2-\nabla^2_{e_i,e_k}f\hat{c}(e_i)c(e_k)+c(e_i)c(\nabla_{e_i}\nabla f)-2\nabla_{\nabla f})\exp(-\epsilon\Box_{Tf}-bL)\right)|_{b=0}+n.p.\\
&=-\epsilon\operatorname{Tr}_s\left((2T|\nabla f|^2-\nabla^2_{e_i,e_k}f\hat{c}(e_i)c(e_k)+\Delta f-2\nabla_{\nabla f})\exp(-\epsilon\Box_{Tf}-bL)\right)|_{b=0}+n.p.
\end{aligned}
\tag{5.32}
$$

Next, we compute

$$
\begin{aligned}
&\frac{\partial}{\partial T}\operatorname{Tr}_s\left(L\exp(-\epsilon\Box_{Tf})\right)+n.p. = -\epsilon\operatorname{Tr}_s\left(L\exp(-\epsilon\Box_{Tf}-b[D_T,\frac{\partial}{\partial T}D_T])\right)|_{b=0}+n.p.\\
&=-\epsilon\operatorname{Tr}_s\left(L\exp(-\epsilon\Box_{Tf}-b[D_T,\hat{c}(df)])\right)|_{b=0}+n.p.\\
&=-\epsilon\operatorname{Tr}_s\left(L\exp(-\epsilon\Box_{Tf}-b\left(c(e_i)\hat{c}(\nabla_{e_i}\nabla f))+2T|\nabla f|^2\right)\right)|_{b=0}+n.p.\\
&=-\epsilon\operatorname{Tr}_s\left(\left(c(e_i)\hat{c}(\nabla_{e_i}\nabla f))+2T|\nabla f|^2\right)\exp(-\epsilon\Box_{Tf}-bL)\right)|_{b=0}+n.p.\\
&=-\epsilon\operatorname{Tr}_s\left(\left(\nabla^2_{e_i,e_k}fc(e_i)\hat{c}(e_k))+2T|\nabla f|^2\right)\exp(-\epsilon\Box_{Tf}-bL)\right)|_{b=0}+n.p.\\
&=-\epsilon\operatorname{Tr}_s\left(\left(-\nabla^2_{e_i,e_k}f\hat{c}(e_k)c(e_i))+2T|\nabla f|^2\right)\exp(-\epsilon\Box_{Tf}-bL)\right)|_{b=0}+n.p.\\
&=-\epsilon\operatorname{Tr}_s\left(\left(-\nabla^2_{e_i,e_k}f\hat{c}(e_i)c(e_k))+2T|\nabla f|^2\right)\exp(-\epsilon\Box_{Tf}-bL)\right)|_{b=0}+n.p..
\end{aligned}
\tag{5.33}
$$

Now let $P_{Tf}(t,x,y)$ be the kernel of $\frac{\partial}{\partial b}\exp(-t\Box_{Tf}+bL)|_{b=0}$.

For fix $\epsilon>0$, integrate by part (we will prove this rigorously in Remark 5.2.10), one has

$$\partial_b \operatorname{Tr}_s \left( (\Delta f - 2\nabla_{\nabla f}) \exp(-\epsilon \Box_{Tf} - bL) \right) |_{b=0} = \int_M (\Delta f - \nabla_{\nabla f}) tr_s(P_{Tf}(\epsilon, x, x)) dvol = 0. \tag{5.34}$$

By (5.32), (5.33) and (5.34), one has

- When $n$ is odd: First, notice that

$$\frac{\partial}{\partial l} Tr_s[N \exp(-t\Box_{t^{-\frac{1}{2}}f})] + n.p. = -t\frac{\partial}{\partial t} Tr_s[L \exp(-t\Box_{Tf})]\big|_{T=t^{-\frac{1}{2}}} + n.p. \tag{5.35}$$

Moreover, by a straightforward computation,

$$\begin{aligned} &-2t\frac{\partial}{\partial t} Tr_s(L \exp(-t\Box_{Tf}))\big|_{T=t^{-\frac{1}{2}}} + n.p. \\ &= -2t\frac{\partial}{\partial t} Tr_s(L \exp(-t\Box_{t^{-\frac{1}{2}}f})) + t^{-\frac{1}{2}}\frac{\partial}{\partial T} Tr_s(L \exp(-t\Box_{Tf}))\big|_{T=t^{-\frac{1}{2}}} + n.p. \end{aligned} \tag{5.36}$$

Now

$$\begin{aligned} \frac{\partial}{\partial l}\psi_{2,1,f}(s) + n.p. &= \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} \frac{\partial}{\partial l} Tr_s(N \exp(-t\Box_{t^{-\frac{1}{2}}f})) dt + n.p. \\ &= -\frac{1}{\Gamma(s)} \int_0^1 t^{s} \frac{\partial}{\partial t} Tr_s(L \exp(-t\Box_{Tf}))\big|_{T=t^{-\frac{1}{2}}} dt + n.p. \\ &= -\frac{1}{\Gamma(s)} \int_0^1 t^{s} \frac{\partial}{\partial t} Tr_s(L \exp(-t\Box_{t^{-\frac{1}{2}}f})) \\ &+ 2t^{s-\frac{3}{2}} \frac{\partial}{\partial l} Tr_s[f \exp(-t\Box_{t^{-\frac{1}{2}}f})] dt + n.p. \text{ (By (5.35) and (5.36))} \end{aligned}$$

Hence, when $s >> 0$

$$\frac{\partial}{\partial l}\psi_{2,1,f}(s)+n.p. = -\frac{s}{\Gamma(s)}\int_0^1 t^{s-1}Tr_s(L\exp(-t\square_{t^{-\frac{1}{2}}f}))dt+Tr_s[L\exp(-\square_f)]-\frac{\partial}{\partial l}\psi_{2,2,f}(s)+n.p..$$

If $n$ is odd, by our previous results in [49], $Tr_s[L\exp(-t\square_{t^{-\frac{1}{2}}f})]+Tr_s[L\exp(-t\square_{-t^{-\frac{1}{2}}f})]$ has no constant term, hence

$$\frac{d}{ds}\frac{\partial}{\partial l}\psi_{2,f}|_{s=0} + n.p. = 0.$$

Consequently,

$$\frac{\partial}{\partial l}R(M, g_l, h, f) = 0.$$

- When $n$ is even, proceed as in the proof of Theorem 4.20 in [20], one can show that $Tr_s[L\exp(-t\square_{t^{-\frac{1}{2}}f})] + Tr_s[L\exp(-t\square_{-t^{-\frac{1}{2}}f})]$ has constant term

$$2\int_M \operatorname{tr}\left[(h_l)^{-1}\frac{\partial h_l^F}{\partial \ell}\right]\frac{(-1)^{\left[\frac{n+1}{2}\right]}}{\pi^{\frac{n}{2}}}\int^B \exp\left(-\frac{\widetilde{R}(x)+|\nabla f|^2-\tilde{\nabla}^2 f(x)}{2}\right)$$
$$-2\int_M \theta_l(F,h_l)\,\tilde{e}_l\left(TM,\nabla^{TM},\nabla_l^{TM},f\right),$$

which finish the proof.

□

**Remark 5.2.10.** *We are going to prove that if $t > 0$ is small enough,*

$$\int_M \Delta f tr_s(\exp(-t\square_{Tf}))dvol = \int_M \nabla_{\nabla f} tr_s(\exp(-t\square_{Td}))dvol.$$

*To this end, fix a bump function $\eta$, s.t. $\eta|_{[0,1]} \equiv 1$, $supp(\eta) \subset [0,2]$. Let $\psi_k(p) := \eta(\frac{|\nabla f|^2(p)}{k})$.*

*By Stoke formula,*

$$\int_M div(\psi_k \nabla f) tr_s(\exp(-t\Box_{Tf}))dvol = \int_M \psi_k \nabla_{\nabla f} tr_s(\exp(-t\Box_{Tf}))dvol. \tag{5.37}$$

*By the short asymptotic expansion of heat kernel, polynomial tameness of $(M, g, f)$ and dominated convergence theorem, let $k \to \infty$, the right hand side of (5.37) goes to*

$$\int_M \nabla_{\nabla f} tr_s(\exp(-t\Box_{Tf}))dvol.$$

*Notice that $|\nabla\psi_k| \leq \frac{C|\nabla^2 f||\nabla f|}{k}$, for the same reason, as $k \to \infty$, the left hand side of (5.37) goes to*

$$\int_M \Delta f tr_s(\exp(-t\Box_{Tf}))dvol.$$

## 5.3 Nine Intermediary Results

Let $(M, g, f)$ be strongly polynomial tame. Let $f : M \to \mathbb{R}$ be a Morse function. Let $\mathrm{Crit}(f)$ be the set of critical points of $f$. If $x \in \mathrm{Crit}(f)$, recall that the index $\mathrm{ind}(x)$ is the number of negative eigenvalues of the quadratic form $d^2 f(x)$ on $T_x M$.

**Definition 5.3.1.** *For $T \geq 1$, let $\mathbb{F}_T^{[0,1]}$ (resp. $\mathbb{F}_T^{(0,1]}$, resp. $\mathbb{F}_T^{\{0\}}$) be the direct sum of the eigenspaces of $\Box_{Tf}$ associated to eigenvalues $\lambda \in [0, 1]$ (resp. $\lambda \in ]0, 1]$, resp. $\lambda = 0$). Let $\Box_{Tf}^{[0,1]}$ (resp. $\Box_{Tf}^{(0,1]}$) be the restriction of $\Box_{Tf}$ to $\mathbb{F}_T^{[0,1]}$ (resp. to $\mathbb{F}_T^{(0,1]}$). For $T \geq 0$, let $P_T^{[0,1]}$ (resp. $P_T^{(0,1]}$, resp. $P_T$) be the orthogonal projection operator from $L^2\Omega^*(M, F)$ on $\mathbb{F}_T^{[0,1]}$ (resp. $\mathbb{F}_T^{(0,1]}$, resp. $\mathbb{F}_T^{\{0\}}$) with respect to the scalar product $\langle\cdot,\cdot\rangle_{\mathbb{F}}$. Here $\langle\cdot,\cdot\rangle_{\mathbb{F}}$ is the inner product induced by $g^{TM}$ and $g^F$. Set $P_T^{(1,+\infty)} = 1 - P_T^{[0,1]}$.*

By Hodge theory, we know that for $0 \leq i \leq n$, $H^i(M, F, d_{Tf}^F)$ and $\mathbb{F}_T^{\{0\},i}$ are canonically isomorphic. As finite dimensional vector subspaces of the $\mathbb{F}_T^i$, the $\mathbb{F}_T^{\{0\},i}$ inherit the scalar product $\langle\cdot,\cdot\rangle_{\mathbb{F}}$. Thus the line $\det H^*(M,F,d^F_{Tf})$ inherits a metric $|\cdot|^{RS}_{\det H^*(M,F,d^F_{Tf})}$, which is also called the $L^2$ metric.

Let $\nabla f \in TM$ be the gradient vector field of $f$. Consider the differential equation

$$\frac{dy}{dt} = -\nabla f(y),$$

which defines a group of diffeomorphism $(\psi_t)_{t\in\mathbb{R}}$ of $M$.

If $x \in \mathrm{Crit}(f)$, set

$$W^u(x) = \left\{y \in M : \lim_{t\to-\infty}\psi_t(y) = x\right\},$$
$$W^s(x) = \left\{y \in M : \lim_{t\to+\infty}\psi_t(y) = x\right\}.$$

The cells $W^u(x)$ and $W^s(x)$ will be called the unstable and stable cells at $x$. We assume that the vector field $\nabla f$ verifies the Smale transversality conditions. Let $x, y \in \mathrm{Crit}(f)$ with $\mathrm{ind}(y) = \mathrm{ind}(x) - 1$. Take $\gamma \in \Gamma(x,y)$. Then $T_yW^u(y)$ is orthogonal to $T_yW^s(y)$ and is oriented. So for $t \in (-\infty,+\infty)$, the orthogonal space $T^{\perp}_{\gamma_t}W^s(y)$ to $T_{\gamma_t}W^s(y)$ in $T_{\gamma_t}M$ carries a natural orientation. Also for $t \in (-\infty,+\infty)$, the orthogonal space $T'_{\gamma_t}W^s(x)$ to $-\nabla f(\gamma_t)$ in $T_{\gamma_t}W^u(x)$ can be oriented in such a way that $s$ is an oriented base of $T'_{\gamma_t}W^u(x)$ if $(-\nabla f(\gamma_t), s)$ is an oriented base of $T_{\gamma_t}W^u(x)$. Finally since $W^u(x)$ and $W^s(y)$ are transversal along $\gamma$, for $t \in (-\infty,+\infty), T^{\perp}_{\gamma_t}W^s(y)$ and $T'_{\gamma_t}W^u(x)$ can be identified, and their orientations can be compared. Set

$$n_\gamma(x,y) = \begin{cases} +1 \text{ if the orientations are the same,} \\ -1 \text{ if the orientations differ.}\end{cases}$$

If $x \in \mathrm{Crit}(f)$, let $[W^u(x)]$ be the real line generated by $W^u(x)$. Let $F$ be a flat vector bundle on $M$, and let $F^*$ be its dual. Set

$$C_\bullet\left(W^u, F^*\right) = \bigoplus_{x\in \mathrm{Crit}(f)} [W^u(x)] \otimes_{\mathbb{R}} F_x^*,$$

$$C_i\left(W^u, F^*\right) = \bigoplus_{\substack{x\in \mathrm{Crit}(f)\\ \mathrm{ind}\ (x)=i}} [W^u(x)] \otimes_{\mathbb{R}} F_x^*.$$

If $x \in \mathrm{Crit}(f)$, the flat vector bundle $F^*$ is canonically trivialized on $W^u(x)$. In particular, if $x, y \in \mathrm{Crit}(f)$ are such that $\mathrm{ind}(y) = \mathrm{ind}(x) - 1$, and if $\gamma \in \Gamma(x, y)$ $s^* \in F_x^*$, let $\tau_\gamma\left(s^*\right) \in F_y^*$ be the parallel transport of $s^* \in F_x^*$ into $F_y^*$ along $\gamma$ with respect to the flat connection of $F^*$.

If $x \in B, s^* \in F_x^*$, set

$$\partial'\left(W^u(x) \otimes f^*\right) = \sum_{\substack{y\in \mathrm{Crit}(f)\\ \mathrm{ind}(y)=\mathrm{ind}(x)-1}} \sum_{\gamma\in\Gamma(x,y)} n_\gamma(x, y) W^u(y) \otimes \tau_\gamma\left(f^*\right).$$

Then $\partial'$ maps $C_i\left(W^u, F^*\right)$ into $C_{i-1}\left(W^u, F^*\right)$.

If $x \in \mathrm{Crit}(f)$, let $[W^u(x)]^*$ be the line dual to the line $[W^u(x)]$. Let $\left(C^\bullet\left(W^u, F\right), \partial\right)$ be the complex which is dual to $\left(C_\bullet\left(W^u, F^*\right), \partial'\right)$. For $0 \le i \le n$, we have the identity

$$C^i\left(W^u, F\right) = \bigoplus_{\substack{x\in \mathrm{Crit}(f)\\ in(x)=i}} [W^u(x)]^* \otimes_{\mathbb{R}} F_x.$$

Then by [25], we know that

$$H^\bullet\left(C^*\left(W^u, F\right), \partial\right) \simeq H^\bullet_{(2)}(M, F, d_f^F),$$

where $H^\bullet_{(2)}(M, F, d_f^F)$ is $F$-valued $L^2$-cohomology.

As a result, we know that

$$\det C^{\bullet}(W^u, F) \simeq \det H^{\bullet}_{(2)}(M, F, d_f^F). \tag{5.38}$$

Now we equip $C^*(W^u)$ with a metric such that for any $x, z \in \mathrm{Crit}(f)$ with $x \neq z$, $W^u(x)^*$ and $W^u(z)^*$ are orthogonal to each other, and that

$$\langle W^u(x)^*, W^u(x)^* \rangle = 1$$

for each $x \in \mathrm{Crit}(f)$. For $x \in \mathrm{Crit}(f)$, let $\|\cdot\|_{\det F_x}$ be a metric on the line $\det F_x$. The metrics $\|\cdot\|_{\det F_x}(x \in \mathrm{Crit}(f))$ induce a metric $\|\cdot\|_{\det C^{\bullet}(W^u,F)}$ on $\det C^{\bullet}(W^u, F)$ .

**Definition 5.3.2.** *The Milnor metric* $\|\cdot\|^{\mathcal{M},\nabla f}_{det\ H^{\bullet}(M,F,d_f^F)}$ *on the line* $\det H^{\bullet}(M, F, d_f^F)$ *is the metric corresponding to the metric* $\|\cdot\|_{\det C^{\bullet}(W^u,F)}$ *via the canonical isomorphism (5.38).*

We now further assume that for any $x \in \mathrm{Crit}(f)$, there exists a sufficiently small open neighborhood $U_x$ of $x$ and a coordinate system $y = (y^1, \dots, y^n.)$ on $U_x$ such that on $U_x$

$$f(y) = f(x) - \frac{1}{2}\left(y^1\right)^2 - \cdots - \frac{1}{2}\left(y^{\mathrm{ind}(x)}\right)^2 + \frac{1}{2}\left(y^{\mathrm{ind}(x)+1}\right)^2 + \cdots + \frac{1}{2}\left(y^n\right)^2,$$

$$g^{TM} = \left(dy^1\right)^2 + \cdots + \left(dy^n\right)^2,$$

$\nabla^F$ preserves the metric $g^F$ in $U_x$. Certainly we can assume that for any $x, y \in \mathrm{zero}(\nabla f)$ with $x \neq y, U_x \cap U_y = \emptyset$ We still assume that $\nabla f$ verifies the Smale transversality conditions. Let $K = \cup_{x \in \mathrm{Crit}(f)} \bar{U}_x$, $\delta_0 := \inf_{p \in M-K} |\nabla f|^2(p)$.

**Theorem 5.3.1.** *The following identity holds,*

$$\lim_{T\to+\infty}\Bigg\{\mathrm{Tr_s}\left[N\log\left(\square_{Tf}^{(0,1]}\right)\right]+\log\left(\frac{|\cdot|^{RS}_{\det H^*(M,F,d^F_{Tf})}}{|\cdot|^{RS}_{\det H^\bullet(M,F,d^F_f)}}\right)^2$$

$$+2\mathrm{rk}(F)\,\mathrm{Tr_s^{Crit}}[f]T+\left(\frac{n}{2}\chi(F)-\tilde{\chi}'(F)\right)\log\left(\frac{T}{\pi}\right)\Bigg\}=\log\left(\frac{\|\cdot\|^{\mathcal{M},\nabla f}_{\det H^\bullet(M,F,d^F_f)}}{|\cdot|^{RS}_{\det H^\bullet(M,F,d^F_f)}}\right)^2.$$

**Theorem 5.3.2.** *Given $\varepsilon, A$ with $0<\varepsilon<A<+\infty$, there exists $C>0$ such that if $t\in[\varepsilon,A], T\geq 1$, then*

$$\left|\mathrm{Tr_s}\left[N\exp\left(-tD_T^2\right)\right]-\tilde{\chi}'(F)\right|\leq\frac{C}{\sqrt{T}}.$$

**Theorem 5.3.3.** *For any $t>0$,*

$$\lim_{T\to+\infty}\mathrm{Tr}_{\mathbf{s}}\left[N\exp\left(-t\square_{Tf}\right)P_T^{(1,+\infty)}\right]=0.$$

*Moreover there exist $c>0$ such that for $t\geq 1, T\geq 0$, then*

$$\left|\mathrm{Tr_s}\left[N\exp\left(-t\square_{Tf}\right)P_T^{(1,+\infty)}\right]\right|\leq c\exp(-t/2).$$

**Theorem 5.3.4.** *For $T\geq 0$ large enough, then*

$$\dim\mathbb{F}_T^{[0,1],i}=\mathrm{rk}(F)M^i,$$

*where $M_i$ is the number of critical points with Morse index $i$.*

*Also*

$$\lim_{T\to+\infty}\mathrm{Tr}\left[\square_{Tf}^{[0,1]}\right]=0.$$

**Theorem 5.3.5.** *As $t \to 0$, the following identity holds,*

$$\mathrm{Tr_s}\left[N\exp\left(-t\square_{t^{-\frac{1}{2}}f}\right)\right] = \mathrm{rk}(F)\int_M\int^B L\exp\left(-\frac{B_1}{2}\right)\frac{1}{\sqrt{t}} + \frac{n}{2}\chi(F) + O(\sqrt{t}),$$

*where for any $T > 0$, $B_T = \frac{\tilde{R}^{TM}}{2} + \sqrt{T}\sum_1^n e^i \wedge \widehat{\nabla_{e_i}^{TM}\nabla} f + T|df|^2 \in \Omega^*(M) \otimes \Omega^*(M)$.*

*Proof.* This follows from Theorem 1.1 and rescaling techniques in [49]. □

**Theorem 5.3.6.** *For any $t > 0$, there is $c > 0$ such that as $T \to +\infty$,*

$$\mathrm{Tr_s}\left[f\exp\left(-t\square_{Tf}\right)\right] = \mathrm{rk}(F)\,\mathrm{Tr_s^{Crit}}[f] + \left(\frac{n}{4}\chi(F) - \frac{1}{2}\widetilde{\chi}'(F)\right)\frac{1}{T} + O\left(e^{-cT}\right).$$

*Proof.* Let $K_T(t, x, y)$ be the heat kernel of $\exp(-t\square_{Tf})$, $c = \delta_0/3$. By Theorem 4.1 in [49] and Proposition 5.2.2, when $T > \frac{1}{t}$, one has

$$\begin{aligned}
|\int_{M-K}\mathrm{tr}_s(fK_T(t,x,x))\mathrm{dvol}(x)| &\le CT^n\int_{M-K}|f||\nabla f|^{2n}\exp(-T|\nabla f|^2)\mathrm{dvol}(x)\\
&\le CT^n\exp(-\frac{T\delta_0}{2})\int_{M-K}|f||\nabla f|^{2n}\exp(-\frac{T|\nabla f|^2}{2})\mathrm{dvol}(x)\\
&\le C'\exp(-cT)
\end{aligned}$$

For the estimation of

$$\left|\int_K \mathrm{tr}_s(fK_T(x,x))\mathrm{dvol}(x) - \mathrm{rk}(F)\,\mathrm{Tr_s^{\mathit{B}}}[f] - \left(\frac{n}{4}\chi(F) - \frac{1}{2}\widetilde{\chi}'(F)\right)\frac{1}{T}\right|,$$

see section 12 in [20]. □

**Theorem 5.3.7.** *For any $d > 0$, there exists $C > 0$ such that for $0 < t \le 1$, $1 \le T \le \frac{d}{t}$, then*

$$\left|\frac{1}{t^2}\left\{\mathrm{Tr_s}\left[f\exp\left(-(tD+T\widehat{c}(\nabla f))^2\right)\right]-\right.\right.$$
$$\left.\left.\mathrm{rk}(F)\int_M f\int^B \exp\left(-B_{T^2}\right)+t\int_M\frac{\theta}{2}\left(F,g^F\right)\int^B \widehat{df}\exp\left(-B_{T^2}\right)\right\}\right|\leq C.$$

*Proof.* Still, this follows from Theorem 1.1 and rescaling techniques in [49]. □

**Theorem 5.3.8.** *For any $T>1$, the following identity holds,*

$$\lim_{t\to 0}\frac{1}{t^2}\left\{\mathrm{Tr_s}\left[f\exp\left(-\left(tD+\frac{T}{t}\widehat{c}(\nabla f)\right)^2\right)\right]-\mathrm{rk}(F)\,\mathrm{Tr_s^{Crit}}[f]\right\}$$
$$=\left(\frac{n}{4}\chi(F)-\frac{1}{2}\widetilde{\chi}'(F)\right)\frac{1}{T\tanh(T)}$$

*Proof.* By a similar arguments in Theorem 5.3.6,

$$|\int_{M-K}\mathrm{tr}_s(fK_{\frac{T}{t^2}}(t^2,x,x))\mathrm{dvol}(x)|\leq C\exp(-\frac{c}{t})$$

as long as $t<\frac{1}{T}$.

For the estimation of

$$\left|\int_K \mathrm{tr}_s(fK_{\frac{T}{t^2}}(t^2,x,x))\mathrm{dvol}(x)-\mathrm{rk}(F)\,\mathrm{Tr_s^B}[f]-t^2\left(\frac{n}{4}\chi(F)-\frac{1}{2}\widetilde{\chi}'(F)\right)\frac{1}{T\tanh(T)}\right|,$$

see Section 14 in [20]. □

**Theorem 5.3.9.** *There exist $c > 0, C > 0$ such that for $t \in (0,1], T \geq 1$, then*

$$\left|\frac{1}{t^2}\left\{\mathrm{Tr_s}\left[f\exp\left(-\left(tD+\frac{T}{t}\widehat{c}(\nabla f)\right)^2\right)\right]-\mathrm{rk}(F)\,\mathrm{Tr_s^{Crit}}[f]\right.\right.$$
$$\left.\left.-\frac{t^2}{T}\left(\frac{n}{4}\chi(F)-\frac{1}{2}\widetilde{\chi}'(F)\right)\right\}\right|\leq C\exp(-cT)$$

*Proof.* By a similar arguments in Theorem 5.3.6,

$$|\int_{M-K}\mathrm{tr}_s(fK_{\frac{T}{t^2}}(t^2,x,x))\mathrm{dvol}(x)|\leq C\exp(-\frac{cT}{t})$$

as long as $t \in (0,1], T \geq 1$.

For the estimation of

$$\left|\int_K \mathrm{tr}_s(fK_{\frac{T}{t^2}}(t^2,x,x))\mathrm{dvol}(x)-\mathrm{rk}(F)\,\mathrm{Tr}_s^B[f]-\frac{t^4}{T}\left(\frac{n}{4}\chi(F)-\frac{1}{2}\widetilde{\chi}'(F)\right)\right|,$$

see Section 15 in [20]. $\square$

## 5.3.1 Proof of Theorem 5.3.1, 5.3.2, 5.3.3 and 5.3.4

if $T > 1$ is large enough, then the map

$$P_{\infty,T}:\mathbb{F}_T^{[0,1]}\longrightarrow C^*\left(W^u,F\right)$$

defined by

$$\omega\otimes s\mapsto\left(\sum_{x\in\mathrm{Crit}(f)}[W^u(x)]^*\int_{W^*(x)}\exp(Tf)\omega\right)\otimes s_x.$$

Without loss of generality we assume that each $U_x, p \in \mathrm{Crit}(f)$, is an open ball around $p$ with radius 4. Let $\gamma : \mathbb{R} \to [0,1]$ be a smooth function such that $\gamma(z) = 1$ if $|z| \leq 1$

and that $\gamma(z) = 0$ if $|z| \geq 2$. For any $x \in \mathrm{Crit}(f)$, set

$$\alpha_{x,T} = \int_{U_x} \gamma(|y|)^2 \exp\left(-T\,|y|^2\right) dvol(y),$$

$$\rho_{x,T} = \frac{\gamma(|y|)}{\sqrt{\alpha_{x,T}}} \exp\left(-\frac{T\,|y|^2}{2}\right) \cdot \rho_x,$$

where $\rho_x = dy^1 \wedge \cdots \wedge dy^{\mathrm{ind}(x)}$.

Let $J_T$ be the unitary map from $C^*\left(W^u, F\right)$ into $\Omega^*(M)$ such that for any $x \in \mathrm{Crit}(f)$ and $T \geq T_0$

$$J_T\left([W^u(x)]^* \otimes s_x\right)(z) = \rho_{x,T} \otimes \tau_\gamma(s_x),$$

where $\gamma$ is a curve connecting $x$ and $z$ in $U_x$, $\tau_\gamma$ is the parallel transform along $\gamma$.

Let $e_T = P_T^{[0,1]} J_T$, then both $e_T : (C^\bullet(W^u, F), \partial) \to (\Omega^*(M), d_{Tf})$ and $J_T : (C^\bullet(W^u, F), \partial) \to (\Omega^*(M), d_{Tf})$ are quasi-isomorphic. Proceed as what we did in [], we have

**Proposition 5.3.3.** *There exists $c > 0$ such that as $T \to +\infty$, for any $\phi \in C^*\left(W^u, F\right)$*

$$\left(e_T - J_T\right)\phi = O\left(e^{-cT(\rho+1)}\right)\|\phi\|_0 \text{ uniformly on } M$$

*In particular, $e_T$ is an isomorphism.*

Let $\mathcal{F} \in \mathrm{End}\left(C^*\left(W^u, F\right)\right)$ which, for $x \in \mathrm{Crit}(f)$, acts on $[W^u(x)]^*$ by multiplication by $f(x)$. Let $\tilde{N} \in \mathrm{End}\left(C^*\left(W^u, F\right)\right)$ which acts on $C^i\left(W^u\right), 0 \leq i \leq n$, by multiplication by $i$. By Proposition 5.3.3, proceed as in [20], one has

**Proposition 5.3.4.** *There exists $c > 0$ such that as $T \to +\infty$,*

$$P_{\infty,T} e_T = e^{T\mathcal{F}} \left(\frac{\pi}{T}\right)^{N/2 - n/4} \left(1 + O\left(e^{-cT}\right)\right).$$

*In particular, $P_{\infty,T}$ is an isomorphism for $T > 0$ large enough.*

By Proposition 5.3.4, proceed as in [20], one can prove Theorem 5.3.1.

Proceed as Proposition 2.4 in [49], one can show that the $k$-th eigenvalue $\lambda_k$ of $\Box_{Tf}$ has at least polynomial growth, i.e., there exists $\alpha_0(n,\alpha,\kappa)>0$, s.t. $\lambda_k \leq C(\delta_f, M, A)(Tk)^{\alpha_0}$, where $\delta_f = \sup_{p\in M} \frac{|Hess(f)|}{1+|\nabla f|^2}$. It's easy to see that $\delta_{Tf} \leq T\delta_f$ when $T \geq 1$.

Hence, by Proposition 5.3.3, Theorem 5.3.2, 5.3.2, 5.3.3 follow.

## 5.4 Cheeger-Muller/Bismut-Zhang Theorem

With nine intermediary results above, proceeding as in Chapter 7 of [20], one has

**Theorem 5.4.1.** *The following identity holds*

$$\log\left(\frac{\|\cdot\|^{RS}_{\det H^\bullet_{(2)}(M,F,d^F_f)}}{\|\cdot\|^{\mathcal{M},\nabla f}_{\det H^\bullet(M,F,d^F_f)}}\right)^2 = -\int_M \theta\left(F,g^F\right)\nabla f^*\psi\left(TM,\nabla^{TM}\right),$$

*where* $\psi = \int_1^{+\infty}\beta_T dT$, $\beta_T = \int^B \frac{\widehat{\nabla f\#}}{2\sqrt{T}}\exp\left(-B_T\right)$, $B_T = \frac{\tilde{R}^{TM}}{2} + \sqrt{T}\sum_1^n e^i \wedge \widehat{\nabla^{TM}_{e_i}\nabla} f + T|df|^2 \in \Omega^*(M)\otimes\Omega^*(M)$.

# Part II

# Landau-Ginzburg B-models and LG/CY Correspondence

# Chapter 6

# LG/CY Correspondence for Weil-Peterson-type Metric and $tt^*$ Structures

In this chapter, we show LG/CY correspondence for $tt^*$ structures and Weil-Peterson type metric.

Let $f : \mathbb{C}^n \to \mathbb{C}$ be a quasi-homogeneous polynomial, i.e., there exist $q_1, \ldots, q_n \in \mathbb{Q}$ such that for any $\lambda \in \mathbb{C}^*$,

$$f(\lambda^{q_1} z_1, \ldots, \lambda^{q_n} z_n) = \lambda f(z_1, ..., z_n).$$

Each $q_i$ is called the weight of $z_i$.

Let $q_i = a_i/b_i$ with $(a_i, b_i) = 1$, and $d = \mathrm{lcm}\,(b_1, \ldots, b_n)$, i.e., $d$ is the least common multiple of $b_1, ..., b_n$. We put $Q_i = q_i d$. Then we say $f$ is quasi-homogeneous with weights $(Q_1, ..., Q_n)$, and has degree $d$.

Let $F(z,u)$ be a marginal deformation of $f$:

$$F(x,u) = f(x) + \sum_{i=1}^{\nu} u^i \psi_i(x),$$

i.e. $l(\psi_i dz_1 ... dz_n) = l(f dz_1 ... dz_n)$, $i = 1, \dots, \nu$ (Recall Definition 1.2.12 for the definition of $l$). We denote by $M$ the space of parameters $u^i$, which should be a small neighborhood of the origin in $\mathbb{C}^l$. Therefore, we have a family of operators $\bar{\partial}_F, \partial_F, \bar{\partial}_F^*, \partial_F^*, \Delta_F$ parameterized by $u \in M$.

For convenience, now we assume that $F(\cdot, u)$ is non-degenerate, i.e. we require that

1. $F(\cdot, u)$ contains no monomial of the form $z_i z_j$ for $i \neq j$,

2. $F(\cdot, u)$ has only an isolated singularity at the origin.

First, we have the following trivial complex Hilbert bundle $L^2\Lambda^*(X) \times M \to M$. For simplicity, denote by $L^2\mathcal{A}$ its $L^2$-integrable section space. There are two natural parings,

$$\begin{aligned} h:\ & L^2\mathcal{A} \times L^2\mathcal{A} \longrightarrow C^\infty(M) \qquad h(\alpha,\beta) = \int_X \alpha \wedge *\bar{\beta}, \\ \eta:\ & L^2\mathcal{A} \times L^2\mathcal{A} \longrightarrow C^\infty(M) \qquad \eta(\alpha,\beta) = \int_X \alpha \wedge *\beta. \end{aligned}$$

In addition, if for differential forms $u, v$, such that $u \wedge *\bar{v}$ is integrable, we still denote

$$h(u,v) := \int_{\mathbb{C}^n} u \wedge *\bar{v}.$$

Moreover, there is a canonical real structure on the Hilbert bundle, which is denoted by $\tau_R$. Then $h(\alpha,\beta) = \eta(\alpha, \tau_R \beta)$

Let $H^n := H^n_F$ be the Hodge bundle over $M$, and its fiber at $u \in M$ is the space of all harmonic $n$-forms of $\Delta_{F(u)}$. We denote the space of its section by $\mathcal{H}$.

Let $\Pi_u : L^2\mathcal{A} \to \mathcal{H}$ be the harmonic projection, $G$ be the inverse operator of $\Delta_F$ on $\mathrm{im}(\bar{\partial}_F) \oplus \mathrm{im}(\bar{\partial}_F^*)$, then $G$ commutes with the operators $\bar{\partial}_F, \partial_F, \bar{\partial}_F^*, \partial_F^*, \Delta_F$, and the operator form of Hodge decomposition reads

$$\mathrm{Id} = \Pi_u + \Delta_F G = \Pi + G\Delta_F.$$

Let $u = (u_1, ..., u_s)$ be local coordinates of $M$, and $\partial_i := \partial_{u_i}$. On the Hodge bundle, we have

(1) The connection $D$, $\bar{D}$

Notice that the Hodge bundle is embedded into the Hilbert bundle, so we can define $D$, $\bar{D}$ in a natural way:

$$D_i = \Pi_u \circ \partial_i, \quad \bar{D}_{\bar{i}} = \Pi_u \circ \bar{\partial}_{\bar{i}} \quad i = 1, ..., s.$$

(2) The operators $C_i, \bar{C}_{\bar{i}}$

We define $C_i = \Pi_u \circ \partial_i F = \Pi_u \circ \psi_i$, $\bar{C}_{\bar{i}} = \Pi_u \circ \overline{\partial_i F} = \Pi_u \circ \overline{\psi_i}$. We can also compute that

$$C_i = (\partial_i F) - \bar{\partial}_F \bar{\partial}_F^* G(\partial_i F), \quad \bar{C}_{\bar{i}} = (\overline{\partial_i F}) - \partial_F \partial_F^* G(\overline{\partial_i F}).$$

By definition, $\bar{C}_{\bar{i}}$ is the adjoint operator of $C_i$ with respect to the $tt^*$ metric $h$, i.e.

$$h(C_i\alpha, \beta) = h(\alpha, \bar{C}_{\bar{i}}\beta).$$

**Proposition 6.0.1** ($tt^*$ equation,[37, 23, 14, 51]). *The operators $D_i, \bar{D}_{\bar{j}}$, $C_i, \bar{C}_{\bar{j}}$ satisfy the following equations*

1. $[C_i, C_j] = 0, \quad [\bar{C}_{\bar{i}}, \bar{C}_{\bar{j}}] = 0, \quad [D_i, \bar{C}_{\bar{j}}] = [\bar{D}_{\bar{i}}, C_j] = 0;$

2. $[D_i, C_j] = [D_j, C_i], \quad [\bar{D}_{\bar{i}}, \bar{C}_{\bar{j}}] = [\bar{D}_{\bar{j}}, \bar{C}_{\bar{i}}]$;

3. $[D_i, D_j] = 0, \quad [\bar{D}_{\bar{i}}, \bar{D}_{\bar{j}}] = 0, \quad [D_i, \bar{D}_{\bar{j}}] = -[C_i, \bar{C}_{\bar{j}}]$.

Fix a homogeneous basis $\{\phi_a\}_{a=0}^{\mu-1}$ of $\mathrm{Jac}(f)$, such that $l(A_a) \le l(A_b)$ if $a < b$, and $\phi_0 = 1$. Moreover, assume that $\{\phi_{a_i}\}_{i=0}^{\mu'-1}$ is a basis of $\mathrm{Jac}(f)'$. **If $f$ is homogeneous of degree $n$.** Let $|u|$ be small enough, such that $\{\phi_a\}_{a=0}^{\mu-1}$ is still a basis of $\mathrm{Jac}(F(\cdot, u))$, $\{\gamma_k\}_{k=0}^{\mu}$ constructed in Subsection 1.2.3 is still a basis of $H_n(\mathbb{C}^n, F(\cdot, u)^{-\infty})$.

## 6.1 Weil-Peterson-type Metric on LG Moduli

First, we will show that: for each $A_a = \phi_a dz_1 ... dz_n$, there exists a harmonic form $w_a = w_a(u) \in \ker(\Delta_{F(\cdot,u)})$ (c.f. Proposition 6.1.2), s.t.

$$w_a(u) = A_a + \bar{\partial}_{F(\cdot,u)} \nu_a(u) \tag{6.1}$$

for some $\nu_a$. Moreover, $\nu_a$ has at most polynomial growth.

To do this, we follow the method in [52] to show that

$$H^n(\mathbb{C}^n, \bar{\partial}_{F(\cdot,u)}) \cong H^n_c(\mathbb{C}^n, \bar{\partial}_{F(\cdot,u)}),$$

where $H^n(\mathbb{C}^n, \bar{\partial}_{F(\cdot,u)})$ is the cohomology of smooth complex (i.e. smooth forms with differential $\bar{\partial}_{F(\cdot,u)}$), and $H^n_c(\mathbb{C}^n, \bar{\partial}_{F(\cdot,u)})$ is the cohomology of smooth complex with compact support ( i.e. compactly supported smooth forms with differential $\bar{\partial}_{F(\cdot,u)}$).

Consider the operator

$$V_F = \frac{(\partial F)^*}{|\nabla F|^2} : \quad \mathcal{A}^{p,q}((\mathbb{C}^n)^*) \to \mathcal{A}^{p-1,q}((\mathbb{C}^n)^*),$$

where $(\mathbb{C}^n)^* = \mathbb{C}^n - \{0\}$, $\mathcal{A}^{p,q}(X)$ is the space of smooth $p,q$ forms on $X$. Next, fix a bump function $\rho$ near 0, and let

$$T_\rho : \mathcal{A}^*((\mathbb{C}^n)^*) \to \mathcal{A}_c^*((\mathbb{C}^n)^*), \qquad R_\rho : \mathcal{A}^*((\mathbb{C}^n)^*) \to \mathcal{A}^*((\mathbb{C}^n)^*)$$

be

$$T_\rho(A) = \rho A + (\bar\partial\rho)V_F \frac{1}{1+[\partial, V_F]}(A), \qquad R_\rho(A) = (1-\rho)V_F \frac{1}{1+[\partial, V_F]}(A).$$

It can be checked easily that (c.f. [52])

**Lemma 6.1.1.** $[\bar\partial_F, R_\rho] = 1 - T_\rho$ *as operators on* $\mathcal{A}(\mathbb{C}^n)$. *Moreover, the embedding* $(\mathcal{A}_c(\mathbb{C}^n), \bar\partial_F) \hookrightarrow (\mathcal{A}(\mathbb{C}^n), \bar\partial_F)$ *is a quasi-isomorphism.*

As a result, $T_\rho(A_a)$ is $L^2$-integrable (since it has a compact support), then we set $w_a := \Pi_u(T_\rho(A_a))$. Consequently

$$w_a = \Pi(T_\rho(A_a)) = (\mathrm{Id} - G\Delta_F)T_\rho(A_a) = A_a - \bar\partial_F R_\rho(A_a) - \bar\partial_F G \bar\partial_F^* T_\rho(A_a).$$

Now set $\nu_a = -R_\rho(A_a) - G\bar\partial_F^* T_\rho(A_a)$, we have (6.1).

Moreover, let $S := \Pi \circ T_\rho$, then

**Proposition 6.1.2.** *The map* $S$ *satisfies the following properties:*

1. $S$ *is a* $C^\infty(M)$*-linear map;*
2. $S$ *is well defined, that is,* $S$ *is independent of the choice of the representative in* $C^\infty(M) \otimes \Omega^n_{X\times M/M}/dF \wedge \Omega^{n-1}_{X\times M/M}$ *and also independent of the cut-off function* $\rho$*.*

*Proof.* (1) The $C^\infty(M)$-linearity follows from the definitions of $T_\rho$ and $\Pi_u$.

(2)

- The independence of the choice of the representative:

  Let $A' = A + dF \wedge B$, for some $B \in C^\infty(M) \otimes \Omega^{n-1}_{\mathbb{C}^n \times M/M}$ with polynomial growth. By definition, there exist $n-1$ forms $\nu$ and $\nu'$ with polynomial growth such that

$$w := S(A) = A + \bar{\partial}_F \nu, \quad w' := S(A') = A' + \bar{\partial}_F \nu'.$$

  Moreover, since $B$ is holomorphic in the $\mathbb{C}^n$ direction,

$$w' := S(A') = A + dF \wedge B + \bar{\partial}_F \nu' = A + \bar{\partial}_F (B + \nu').$$

  Then one has

$$w - w' = \bar{\partial}_{F(\cdot,u)}(\nu + B - \nu'),$$

  where $(\nu + B - \nu')$ has polynomial growth. Then by Agmon estimate and integration by parts, one has

$$h(w - w', w - w') = h(w - w', \bar{\partial}_{F(\cdot,u)}(\nu - \nu')) = h(\bar{\partial}^*_{F(\cdot,u)}(w - w'), \nu - \nu') = 0,$$

  which implies that $w = w'$.

- The independence of the choice of the cut-off function:

  Let $S$ and $S'$ be the map with respect to the cut-off functions $\rho$ and $\rho'$ respectively. Then

$$A = S(A) - \bar{\partial}_{F(\cdot,u)}\nu = S'(A) - \bar{\partial}_{F(\cdot,u)}\nu'$$

  for some differential forms $\nu$ and $\nu'$ with polynomial growth. Then repeat the same argument as above, one shows that $S(A) = S'(A)$ easily.

$\square$

**Proposition 6.1.3.** *The section $w_a(u)$ is a holomorphic section, i.e. $\bar{D}_{\bar{i}} w_a = 0$*

*Proof.* Let $\bar{\partial}_{\bar{i}} := \frac{\partial}{\partial \bar{u}_i}$, then it's straightforward that $[\bar{\partial}_{\bar{i}}, \bar{\partial}_{F(\cdot,u)}] = 0$. Hence

$$\bar{\partial}_{\bar{i}} w_a(u) = \bar{\partial}_{\bar{i}} A_a - \bar{\partial}_{\bar{i}} \bar{\partial}_{F(\cdot,u)} \nu_a = -\bar{\partial}_{\bar{i}} \bar{\partial}_{F(\cdot,u)} \nu_a = -\bar{\partial}_{F(\cdot,u)} \bar{\partial}_{\bar{i}} \nu_a.$$

Following the same arguments as in the proof of Proposition 6.1.2, one shows that for any harmonic form $w$, $h(\bar{\partial}_{\bar{i}} w_a, w) = 0$. As a result,

$$\bar{D}_{\bar{i}} w_0(u) = \Pi_u \bar{\partial}_{\bar{i}} w_a(u) = 0.$$

$\square$

### 6.1.1 $U(1)$ Actions on $\mathcal{A}(\mathbb{C}^n)$

$\mathbb{C}^n$ admits a unitary $U(1)$ action associated with $f$, which is given by

$$e^{i\theta} \cdot (z_1, \ldots, z_n) = \left(e^{i\theta Q_1} z_1, \ldots, e^{i\theta Q_n} z_n\right).$$

In particular, if $f$ is homogeneous,

$$e^{i\theta} \cdot (z_1, \ldots, z_n) = \left(e^{i\theta} z_1, \ldots, e^{i\theta} z_n\right).$$

This action induces a unitary action $\mathcal{T}$ of $U(1)$ on differential forms, such that for any $p, q$ form $\alpha = \alpha(z_1, \bar{z}_1 ..., z_n, \bar{z}_n) dz_{i_1} ... dz_{i_p} d\bar{z}_{j_1} ... d\bar{z}_{j_q}$,

$$\mathcal{T}(\theta)\alpha = e^{i\theta(-\sum_{k=1}^p Q_{i_k} + \sum_{k=1}^q Q_{j_k})} \alpha(e^{-iQ_1\theta} z_1, e^{iQ_1\theta} \bar{z}_1, ..., e^{-iQ_n\theta} z_n, e^{iQ_n\theta} \bar{z}_n) dz_{i_1} ... dz_{i_p} d\bar{z}_{j_1} ... d\bar{z}_{j_q}.$$

In particular, if $f$ is homogeneous,

$$\mathcal{T}(\theta)\alpha = e^{i(q-p)\theta}\alpha(e^{-i\theta}z_1, e^{i\theta}\bar{z}_1, ..., e^{-i\theta}z_n, e^{i]\theta}\bar{z}_n)dz_{i_1}...dz_{i_p}d\bar{z}_{j_1}...d\bar{z}_{j_q}.$$

Now consider the following $U(1)$ action $\mathcal{P}$ on $L^2\Lambda^{p,q}(\mathbb{C}^n)$,

$$\mathcal{P}(\theta)(\alpha) = e^{idp\theta}\mathcal{T}(\theta)\alpha, \quad \alpha \in L^2\Lambda^{p,q}(\mathbb{C}^n).$$

Similarly, we have another $U(1)$ action $\mathcal{Q}(\theta) := e^{idq\theta}\mathcal{T}(-\theta)$ on $L^2\Lambda^{p,q}(\mathbb{C}^n)$, where $d$ is the degree of quasi-homogeneous polynomial $f$.

**Remark 6.1.4.** *One can see later that the infinitesimal action of $\mathcal{P}$ and $\mathcal{Q}$ plays the roles of $p, q$ degree in LG models. Comparing with CY's case, let $X$ be a compact Calabi-Yau manifold, on $\mathcal{A}^{p,q}(X)$, one defines $\mathcal{P}^{CY}(\theta)w = e^{ip\theta}w$, $\mathcal{Q}^{CY}(\theta)w = e^{iq\theta}w$ for $w \in A^{p,q}(X)$.*

Immediately, we have,

**Lemma 6.1.5.** *The $U(1)$ action $\mathcal{P}$ and $\mathcal{Q}$ satisfy the following properties:*

*(1) $\mathcal{P}$ and $\mathcal{Q}$ are unitary, i.e.,*

$$h(\mathcal{Q}(\theta)\alpha, \mathcal{Q}(\theta)\beta) = h(\mathcal{P}(\theta)\alpha, \mathcal{P}(\theta)\beta) = h(\alpha, \beta), \quad \alpha, \beta \in L^2\Lambda^{p,q}(\mathbb{C}^n).$$

*(2) $[\mathcal{P}(\theta), \bar{\partial}_F] = 0$, $[\mathcal{P}(\theta), \bar{\partial}_F^*] = 0$, $[\mathcal{P}(\theta), \Delta_F] = 0$.*

*(3) $\mathcal{P}(\theta)\partial_F\mathcal{P}(\theta)^{-1} = e^{id\theta}\partial_F$, $\mathcal{P}(\theta)\partial_F^*\mathcal{P}(\theta)^{-1} = e^{-id\theta}\partial_F^*$.*

*(4) $[\mathcal{Q}(\theta), \partial_F] = 0$, $[\mathcal{Q}(\theta), \partial_F^*] = 0$, $[\mathcal{Q}(\theta), \Delta_F] = 0$.*

*(5) $\mathcal{Q}(\theta)\bar{\partial}_F\mathcal{Q}(\theta)^{-1} = e^{id\theta}\bar{\partial}_F$, $\mathcal{P}(\theta)\bar{\partial}_F^*\mathcal{P}(\theta)^{-1} = e^{-id\theta}\bar{\partial}_F^*$.*

*Moreover, (2) and (4) imply that $\mathcal{P} : \mathcal{H} \to \mathcal{H}$, $\mathcal{Q} : \mathcal{H} \to \mathcal{H}$.*

**Remark 6.1.6.** *In CY's case, one can also see that $\mathcal{P}^{CY}, \mathcal{Q}^{CY}$ are unitary; $\mathcal{P}^{CY}$ commutes with $\bar{\partial}$; $\mathcal{P}^{CY}(\theta)\partial\mathcal{P}^{CY}(\theta)^{-1} = e^{i\theta}\partial$.*

Recall that

**Definition 6.1.7.** *For $A = z_1^{\beta_1} \cdots z_n^{\beta_n} dz_1 \wedge \cdots \wedge dz_n$, we define*

$$l(A) := \sum_{i=1}^{n} (\beta_i + 1) Q_i.$$

*Moreover, let $l_a = l(A_a)$.*

**Lemma 6.1.8.** *The $U(1)$ action $\mathcal{P}$ and $\mathcal{Q}$ commute with the map $S$, that is*

$$S(\mathcal{P}(\theta)A) = \mathcal{P}(\theta)(S(A)), \quad A \in C^\infty(M) \otimes \Omega^n / dF(\cdot, u) \wedge \Omega^{n-1}$$

*and*

$$S(\mathcal{Q}(\theta)A) = \mathcal{Q}(\theta)(S(A)), \quad A \in C^\infty(M) \otimes \Omega^n / dF(\cdot, u) \wedge \Omega^{n-1}.$$

*More explicitly,*

$$\mathcal{P}(\theta)(w_a) = e^{i(nd - l(A_a))\theta} w_a,$$

$$\mathcal{Q}(\theta)(w_a) = e^{i(l(A_a))\theta} w_a.$$

*Thus, we could define the $U(1)$ charge of $w_a$ to be $l_a$. This lemma could be rephrased as follows: if a harmonic form $w$ has $U(1)$ charges $l$, that is, $w = S(A)$ for some homogeneous $n$-form $A$ with $l(A) = l$, then*

$$\mathcal{P}(\theta)(w) = e^{i(nd-l)\theta} w,$$

$$\mathcal{Q}(\theta)(w) = e^{il\theta} w.$$

**Remark 6.1.9.** *If $f$ contains a unique isolated singularity at the origin, the strong nullstellensatz implies the existence of a sufficiently large $N$ such that $z_i^N \in \langle \partial_{z_1} f, ..., \partial_{z_n} f\rangle$ for all $i = 1, 2, ..., n$. As a result, if $\phi$ is homogeneous with a sufficient large degree $D$, $\phi \in \langle \partial_{z_1} f, ..., \partial_{z_n} f\rangle$. In fact, lemma 6.1.5 could provide an explicit description of $D$. To keep things simple, we will consider only the case where $f$ satisfies the Calabi-Yau condition. A harmonic form's $U(1)$ charge cannot exceed $n(n-1)$ in this case. Thus, if $w = S(\phi dz_1...dz_n)$ for some homogeneous polynomial $\phi$ of degree $n(n-2)+1$, $w$ must have a $U(1)$ charge of $n(n-1)+1$. As a result, $w$ should be trivial, implying that $\phi \in \langle \partial_{z_1} f, ..., \partial_{z_n} f\rangle$. As a result, we have $D = n(n-2)+1$ in this case.*

*Proof.* It suffices to check the result for $A_a = \phi_a dz_1 \wedge \cdots \wedge dz_n$. First, notice that

$$\mathcal{P}(\theta)A_a = e^{i(nd-l(A_a))\theta}A_a. \tag{6.2}$$

Since

$$A_a = S(A_a) - \bar{\partial}_{F(\cdot,u)}\nu_a, \tag{6.3}$$

by Lemma 6.1.5, we have

$$\mathcal{P}(\theta)A_a = \mathcal{P}(\theta)S(A_a) - \bar{\partial}_F(\mathcal{P}(\theta)\gamma_a); \tag{6.4}$$

then multiply $e^{i(n^2-l_a)\theta}$ on the both sides of (6.3), one has

$$e^{i(nd-l(A_a))\theta}A_a = e^{i(nd-l(A_a))\theta}S(A_a) - \bar{\partial}_F\left(e^{i(nd-l(A_a))\theta}\gamma_a\right). \tag{6.5}$$

(6.2), (6.4) and (6.5) imply that

$$e^{i(nd-l(A_a))\theta}S(A_a) - \mathcal{P}(\theta)S(A_a) = \bar{\partial}_F\mu_a$$

for some differential form $\mu_a$ with at most polynomial growth.

By Agmon estimate and the fact that $[\mathcal{P}(\theta), \Delta_f] = 0$ , we get

$$\mathcal{P}(\theta)S(A_a) = e^{i(nd-l(A_a))\theta}S(A_a) = S(\mathcal{P}(\theta)A_a).$$

□

In CY's case, if $u \in \mathcal{A}^{p,n-p}(X), v \in \mathcal{A}^{p',n-p'}(X)$, then $\int_X u \wedge *\bar{v} \neq 0$ iff $p = p'$. Similarly, by unitary property of $\mathcal{P}$ and Lemma 6.1.8, it is easy to obtain

**Corollary 6.1.1.** *If $h(w_a, w_b) \neq 0$, then $l(A_a) = l(A_b)$.*

*Proof.* Just notice that, by Lemma 6.1.8,

$$h(w_a, w_b) = h(\mathcal{P}(\theta)w_a, \mathcal{P}(\theta)w_b) = h(e^{i(nd-l_a)\theta}w_a, e^{i(nd-l_b)\theta}w_b) = e^{i(l_b-l_a)\theta}h(w_a, w_b).$$

Hence, if $h(w_a, w_b) \neq 0$, we must have $l_b = l_a$. □

**Definition 6.1.10.** *We define the bi-grading $\hat{p}, \hat{q} : \mathcal{H} \to \mathbb{Q}$ as follows,*

$$\hat{p}(w_a) = n - \frac{l_a}{d} - \sum_{i=1}^{n} q_i,$$

$$\hat{q}(w_a) = \frac{l_a}{d} - \sum_{i=1}^{n} q_i.$$

*Then one can see that, restricted to $\mathcal{H}$,*

$$\frac{d}{d\theta}\mathcal{P}(\theta)|_{\theta=0} = d \times \hat{p}(w_a) + \sum_{i=1}^{n} Q_i$$

*and*

$$\frac{d}{d\theta}\mathcal{Q}(\theta)|_{\theta=0} = d \times \hat{q}(w_a) + \sum_{i=1}^{n} Q_i,$$

Then Corollary 6.1.1 could be interpreted as if $\hat{p}(w_a) \neq \hat{p}(w_b)$, then $h(w_a, w_b) = 0$.

Let $h_{0\bar{0}} = h(w_0, w_0)$,

$$0 = h_{0\bar{a}} = h(w_0, w_a),$$

then by Proposition 6.1.3

$$\partial_i h(w_0, w_0) = h(D_i w_0, w_0) + h(w_0, \bar{D}_{\bar{i}} w_0) = h(D_i w_0, w_0),$$

$$0 = \partial_i h(w_0, w_a) = h(D_i w_0, w_a) + h(w_0, \bar{D}_{\bar{i}} w_a) = h(D_i w_0, w_a).$$

Hence, we have

$$D_i w_0 = \left(h_{\bar{0}0}^{-1} \partial_i h_{0\bar{0}}\right) w_0, \tag{6.6}$$

which means that we have a line subbundle $\mathcal{L}$ of $\mathcal{H}^n$ with connection $D + \bar{D}$(Comparing with $\mathcal{L}^{CY}$ and $D^{CY} + \bar{D}^{CY}$ in Section 2.5).

Similarly, for each $w_a = S(A_a)$, $D_i w_a$ is a linear combination of $\{w_b\}$ with $l_b = l_a$, i.e. $D$ and $\bar{D}$ preserve the bi-grading $(\hat{p}, \hat{q})$

In Section 2.5, one can see that $C^{CY}$ shifts the bi-grading $(p, q) \to (p-1, q+1)$, and $\bar{C}^{CY}$ shifts the bi-grading $(p, n-p) \to (p+1, q-1)$. Similarly, one can check easily that $\mathcal{P}(\theta)(C_i w_a) = e^{i(nd-l_a-d)\theta} w_a$; and correspondingly, $\mathcal{P}(\theta)(\bar{C}_i w_a) = e^{i(nd-l_a+d)\theta} w_a$. Hence $C$ shiftes the bi-grading $(\hat{p}, \hat{q}) \to (\hat{p}-1, \hat{q}+1)$, and $\bar{C}$ shiftes the bi-grading $(\hat{p}, \hat{q}) \to (\hat{p}+1, \hat{q}-1)$. Again, since $w_0$ is the unique section with the fiberwise lowest $\hat{q}$-grading (up to a multiplication of $C^\infty(M)$), we have

$$\bar{C}_{\bar{i}} w_0 = 0. \tag{6.7}$$

*Proof of Theorem 1.3.8.* First, by Proposition 6.1.3, one computes

$$\partial_i \log(h(w_0, w_0)) = \frac{h(D_i w_0, w_0)}{h(w_0, w_0)}.$$

Then

$$\begin{aligned}
\bar{\partial}_{\bar{j}} \partial_i \log(h(w_0, w_0)) &= \frac{h(\bar{D}_{\bar{j}} D_i w_0, w_0) + h(D_i w_0, D_j w_0)}{h(w_0, w_0)} - \frac{h(D_i w_0, w_0) h(w_0, D_j w_0)}{h(w_0, w_0)^2} \\
&= \frac{h(\bar{D}_{\bar{j}} D_i w_0, w_0)}{h(w_0, w_0)} \text{ ( By (6.6))} \\
&= \frac{h([\bar{D}_{\bar{j}}, D_i] w_0, w_0)}{h(w_0, w_0)} \text{ ( By Proposition 6.1.3)} \\
&= -\frac{h([\bar{C}_{\bar{j}}, C_i] w_0, w_0)}{h(w_0, w_0)} \text{ ( By Proposition 6.0.1)} \\
&= \frac{h(C_i w_0, C_j w_0)}{h(w_0, w_0)} \text{ ( By (6.7)).} \\
&= \frac{h(w_i, w_j)}{h(w_0, w_0)}.
\end{aligned}$$

□

## 6.2 Real Structures

**Now assume that $f$ is homogeneous of degree $n$, i.e. $Q_i \equiv 1$, $d = n$.**

In this case,

$$\hat{p}(w_a) = n - \frac{l_a}{n} - 1,$$

$$\hat{q}(w_a) = \frac{l_a}{n} - 1.$$

Using the isomorphism $S$, we define a real structure $\kappa$ on $C^\infty(M)\Omega^n / dF(\cdot, u) \wedge \Omega^{n-1}$ :

Since $\tau_R(w_a) = \bar{w}_a \in \mathcal{H}$, there exists a $\mu \times \mu$ matrix $M(u, \bar{u})$, s.t.

$$\tau_R(w_a) := \bar{w}_a = M_{\bar{a}}^b(u, \bar{u}) w_b,$$

then we define an anti-linear map $\kappa$ by

$$\kappa(A_a) := M_{\bar{a}}^b(u, \bar{u}) A_b. \tag{6.8}$$

One can check easily that if $w_a = S(A_a)$, then $\bar{w}_a = S(\kappa(A))$, moreover:

**Proposition 6.2.1.** *Let $A \in C^\infty(M) \otimes \Omega^n / df \wedge \Omega^{n-1}$ has homogeneous degree $l(A)$, then $\kappa(A)$ is also homogeneous with degree $l(\kappa(A))$. Furthermore,*

$$l(A) + l(\kappa(A)) = n^2.$$

*Proof.* It suffices to prove it for $\{A_a\}_{a=0}^{\mu-1}$.

Let $w_a = S(A_a)$, i.e.

$$w = A + \bar{\partial}_F \nu_a \quad \text{for some } (n-1)\text{-form } \nu_a (\text{with polynomial growth}).$$

Then

$$\bar{w}_a = \bar{A}_a + \partial_F \bar{\nu}_a. \tag{6.9}$$

First by Lemma 6.1.5, (6.9), and $\mathcal{P}(\theta)\partial_F = e^{i\theta}\partial_F \mathcal{P}(\theta)$, proceeding as what we did in Lemma 6.1.8, one has

$$\mathcal{P}(\theta)\bar{w}_a = e^{il_a\theta}\bar{w}_a = e^{il_a} \sum_b M_a^b w_b. \tag{6.10}$$

On the other hand,

$$\bar{w}_a = \kappa(A_a) + \bar{\partial}_F \tilde{\nu}_a \quad \text{for some } (n-1)\text{-form } \tilde{\nu}_a \text{ (with polynomial growth),}$$

and $\kappa(A_a) = \sum_b M(u,\bar{u})_a^b A_b$. Hence

$$\mathcal{P}(\theta)\bar{w}_a = \sum_b e^{i(n^2-l_b)\theta} M_a^b w_b. \tag{6.11}$$

(6.10)$-$(6.11), one has

$$\sum_b (e^{i(n^2-l_b)} - e^{il_a}) M_a^b w_b = 0.$$

Since $\{w_a\}$ is a basis, one has $M_a^b = 0$ if $n^2 - l_b \neq l_a$, which implies that $\kappa(A_a)$ is homogeneous of degree $n^2 - l_a$. □

One can also interpret Proposition 6.8 as $\hat{p}(w_a) = \hat{q}(\bar{w}_a)$.

Hence, the small $tt^*$ structure for LG model is well defined:

**Proposition/Definition 6.2.2** (small $tt^*$ structure in the Landau-Ginburg B models [23])**.** *Let $f$ be a quasi-homogeneous polynomial and $F$ be its marginal deformation with parameter space $M$. Let $\mathcal{H}' \subset \mathcal{H}$ be the subbundle generated by $w_{k_a} = S(A_{k_a})$, where $l(A_{k_a})/n \in \mathbb{Z}$. By Proposition 6.2.1, restriction of $\tau_R, h, D, C, \bar{C}$ to $\mathcal{H}'$ defines a $tt^*$ structure, called small $tt^*$ geometry structure in the Landau-Ginzburg B models.*

*Proof.* The restriction gives a well defined $tt^*$ structure, since

(1) $D$ preserves the bi-grading of each $w_a$:

Since $w_a$ is holomorphic, i.e. $\bar{D}_{\bar{i}} w_a = 0$, we have

$$\partial_i h(w_a, w_b) = h(D_i w_a, w_b) + h(w_a, \bar{D}_{\bar{i}} w_b) = h(D_i w_a, w_b).$$

By the non-degeneracy of $h$ and Corollary 6.1.1, we can see that $D$ and $\bar{D}$ preserve the bi-grading.

(2) $C_i$ shifts the bi-grading $(\hat{p}, \hat{q}) \to (\hat{p}-1, \hat{q}+1)$.

(3) $\mathcal{H}'$ is stable under $\tau_R$ by Proposition 6.2.1. Combining with (1) and (2), (3) also implies that $\mathcal{H}'$ is stable under $\bar{D}_{\bar{i}}$ and $\bar{C}_{\bar{i}}$.

□

## 6.3 Residue Maps

Set $\hat{q}_a = \hat{q}(w_a)$. Then for each $\phi_a \in \mathrm{Jac}(f)'$, $\hat{p}_a \in \mathbb{Z}^+$. Let $\Omega := \sum_{i=1}^n (-1)^{i+1} z_i dz_1 \wedge \ldots \widehat{dz_i} \ldots dz_n$, and denote $\Omega_a := \frac{\phi_a \Omega}{f^{n_a}}$. Then one can see that $\Omega_a \in H^{n-1}(\mathbb{C}P^{n-1} - X_f)$, where $X_f := \{[z] \in \mathbb{C}P^{n-1} : f(z) = 0\}$. Let $\mathrm{res} : H^{n-1}(\mathbb{C}P^{n-1} - X_f) \to H^{n-2}(X_f)$ be the residue map, then it was shown that (c.f. [21])

$$\mathrm{res}(\Omega_a) \in F^{\hat{p}_a} H^{n-2}(X_f)_0, \tag{6.12}$$

where $H^{n-2}(X_f)_0$ denotes the space of primitive forms in $H^{n-2}(X_f)$. In particular, $\mathrm{res}(\Omega_0)$ is a nowhere vanishing holomorphic $(n-2, 0)$ form on $X_f$.

Moreover, let $\tau : H_{n-2}(X_f) \to H_{n-1}(\mathbb{C}P^{n-1} - X_f)$ be the Leray coboundary map, then for any cycle $\delta \in H_{n-2}(X_f)$,

$$2\pi i \int_\delta \mathrm{res}(\Omega_a) = \int_{\tau(\delta)} \Omega_a.$$

**Definition 6.3.1.** *We define the map* $: \mathrm{Jac}(F)' \to H^{n-2}(X_F)$ *via*

$$R(\phi_a) := \mathrm{res}(\frac{\phi_a \Omega}{F^{n_a}}).$$

*In particular, $R(1)$ is a nowhere vanishing holomorphic $(n-2,0)$ form on $X_F$.*

## 6.4 Intersection Matrix and Riemann Bilinear Formula

Before moving on, let's recall the construction of $\gamma_k$: Let $c_t(s) := te^{2\pi i s}$, $0 \le s \le 1$, then $c_t$ induces a monodromy operator $M : H_{n-1}(V_t) \to H_{n-1}(V_t)$. Following [23], we fix a basis $\{\sigma_k\}_{k=0}^{\mu-1}$ of $H_{n-1}(V_{-1})$, such that $\sigma_k \in \ker(M - \mathrm{Id})$ for $0 \le k \le \mu' - 1$. Hence, there exists $\delta_k \in H_{n-2}(V_\infty)_0 := (H^{n-2}(V_\infty)_0)^*$, s.t. $\sigma_k = \tau(\delta_k)$ for $0 \le k \le \mu' - 1$. For $t > 0$, let $\Phi_t : V_{-t} \to V_{-1}$ be the map $(z_1, .., z_n) \to (t^{-\frac{1}{n}} z_1, ..., t^{-\frac{1}{n}} z_n)$, and set

$$\gamma_k := \cup_{t>0} (\Phi_t)^* \sigma_k,$$

then one can see easily that $\{\gamma_k\}_{k=0}^{\mu-1}$ is a basis of $H_n(\mathbb{C}^n, f^{-\infty})$ (called Lefchetz thimble).

Similarly, one can construct a basis $\{\tilde{\gamma}_k\}_{k=0}^{\mu-1}$ of $H_n(\mathbb{C}^n, f^{+\infty})$.

Now we would like to define intersection matrix, first, we show that:

For each $\gamma_k \in H_n(\mathbb{C}^n, F^{-\infty})$, $w \mapsto \int_{\gamma_k} e^{F+\bar{F}} w$ is linear for all $w \in \ker \Delta_{F(\cdot,u)}$, hence by Riesz representation, there exists $\alpha_k \in \ker \Delta_{F(\cdot,u)}$, s.t.

$$\int_{\gamma_k} e^{F+\bar{F}} w = \int_{\mathbb{C}^n} w \wedge *\alpha_k = \int_{\mathbb{C}^n} e^{F+\bar{F}} w \wedge e^{-F-\bar{F}} * \alpha_k.$$

Then set $PD(\gamma_k) := e^{-f-f} * \alpha_k \in H^n(\mathbb{C}^n, F^{+\infty})$. Similarly, we could define $PD(\tilde{\gamma}_k)$, such that

$$\int_{\tilde{\gamma}_k} e^{-F-\bar{F}} * w = \int_{\mathbb{C}^n} PD(\tilde{\gamma}_k) \wedge e^{-F-\bar{F}} * w$$

for all $w \in \ker \Delta_{F(\cdot,u)}$.

**Definition 6.4.1.** *The intersection matrix $I$ is defined by*

$$I_{ij} = \int_{\gamma_i} PD(\tilde{\gamma}_j) = \int_{\mathbb{C}^n} PD(\tilde{\gamma}_j) \wedge *PD(\gamma_i).$$

*for $0 \leq i, j \leq \mu - 1$.*

*While $I'$ is a submatrix of $I$, that is $I'_{ij} = I_{ij}$ for $0 \leq i, j \leq \mu' - 1$. Now set $\mathcal{I} = I^{-1}, \mathcal{I}' = (I')^{-1}$.*

From the definition, it seems that $PD(\gamma_k)$ depends on $u$, however,

**Proposition 6.4.2.** *$I$ is locally constant, i.e. when $|u|$ is small, $\partial I = \bar{\partial} I = 0$.*

*Proof.* Suppose $w$ is $\Delta_{F(\cdot,u)}$-harmonic, then if $|u|$ is small, by Agmon estimate, $w' := e^{(F(\cdot,u)-f)+\overline{F(\cdot,u)-f}} w$ has exponential decay and is $d_{2\mathrm{Re}(f)}$-closed. Then proceed as what we did in Section 7.5 of [25], we can find $\nu$, s.t. $w' - \Pi_0 w' = d_{2\mathrm{Re}(f)}\nu$, and $\nu$ has exponential decay. Here $\Pi_u$ is the projection $L^2 \to \ker \Delta_{F(\cdot,u)}$. Hence,

$$\int_{\gamma_k} e^{F+\bar{F}} w = \int_{\gamma_k} e^{f+\bar{f}} w' = \int_{\gamma_k} e^{f+\bar{f}} \Pi_0 w'.$$

Again, proceeding as what we did in Section 7.5 of [25], $\Pi_0 \circ e^{(F(\cdot,u)-f)+F(\cdot,\bar{u})-f} : H^*_{(2)}(\mathbb{C}^n, d_{F(\cdot,u)}) \to H^*_{(2)}(\mathbb{C}^n, d_f)$ is isomorphic. For a similar reason, one can see that $PD(\gamma_k)(u,\bar{u}) - PD(\gamma_k)(0) = d\mu$ for some differential form $\mu$ with exponential decay on $\mathrm{Re}(f) < 0$.

Hence, $I$ is locally constant. □

By the definition of intersection matrix, the following Riemann bilinear formula is straightforward:

**Proposition 6.4.3.** *If $w$ and $w'$ are harmonic, then*

$$\int_{\mathbb{C}^n} w \wedge *w' = \sum_{k,l} \left( \int_{\gamma_k} e^{F+\bar{F}} w \right) (\mathcal{I})_{kl} \left( \int_{\tilde{\gamma}_l} e^{-F-\bar{F}} w' \right).$$

Actually, one also have

**Proposition 6.4.4.** *If $w$ is harmonic,*

$$\int_{\mathbb{C}^n} e^{-\bar{F}} A_a \wedge *w = \sum_{k,l} \left( \int_{\gamma_k} e^F A_a \right) (\mathcal{I})_{kl} \left( \int_{\tilde{\gamma}_l} e^{-F-\bar{F}} w \right).$$

We will prove this proposition in the appendix.

## 6.5 Period Integrals

Next, we would like to study period integrals on LG models. There are two different notions of period integrals. This section is dedicated to proving Theorem 1.3.9.

Firstly, by [25] or Theorem 1.3.3, one can see that $\{e^{F+\bar{F}} w_a\}$ ($\{e^{-F-\bar{F}} * w_a\}$) is a basis of $H^n(\mathbb{C}^n, F(\cdot, u)^{-\infty})$ ($H^n(\mathbb{C}^n, F(\cdot, u)^{+\infty})$). Consequently, the matrix

$$(\mathcal{A})_{ka} := \int_{\gamma_k} e^{F+\bar{F}} w_a$$

and

$$(\tilde{\mathcal{A}})_{ka} := \int_{\tilde{\gamma}_k} e^{-F-\bar{F}} * w_a$$

are invertible.

Secondly,

$$(\mathcal{B})_{ka} := \int_{\gamma_k} e^F A_a,$$

and

$$(\tilde{\mathcal{B}})_{ka} := \int_{\gamma_k} e^{-F} * A_a.$$

As a result, there exist matrix $\mathcal{T}$ and $\tilde{\mathcal{T}}$, such that $\mathcal{B} = \mathcal{T}\mathcal{A}$ and $\tilde{\mathcal{B}} = \tilde{\mathcal{T}}\tilde{\mathcal{A}}$. More specifically,

$$\int_{\gamma_k} e^F A_a = \sum_b (\mathcal{T})_{ab} \int_{\gamma_k} e^{F+\bar{F}} w_b$$

*Proof of Theorem 1.3.9.* By Riemann bilinear formula, on the one hand,

$$h(e^{-F}A_a, w_c) = \left(\int_{\gamma_k} e^F A_a\right) \mathcal{I}_{kl} \overline{\left(\int_{\tilde{\gamma}_l} e^{-F-\bar{F}} * w_c\right)}. \tag{6.13}$$

On the other hand,

$$\left(\int_{\gamma_k} e^F A_a\right) \mathcal{I}_{kl} \overline{\left(\int_{\tilde{\gamma}_l} e^{-F-\bar{F}} * w_c\right)} = \sum_{b:l(A_c)=l(A_b)} (\mathcal{T})_{ab} h(w_b, w_c). \tag{6.14}$$

Let $t = e^{i\theta}$, then

$$h(e^{-\bar{F}}A_a, w_c) = h(\mathcal{P}(\theta)e^{-\bar{F}}A_a, \mathcal{P}(\theta)w_c) = t^{l_c-l_a} h(e^{-\bar{t}^{-n}\bar{F}}A_a, w_c). \tag{6.15}$$

- If $\frac{l_c-l_a}{n} \notin \mathbb{Z}$, then take $t$ to be the $n$-th root of 1, such that $t^{l_c-l_a} \neq 1$. By (6.15), we can see that

  $$h(e^{-\bar{F}}A_a, w_c) = 0.$$

  This together with (6.13), (6.14), and the fact that the matrix $(h(w_b, w_c))_{l_b=l_c}$ is invertible implies that $(\mathcal{T})_{ab} = 0$ if $\frac{l_a-l_b}{n} \notin \mathbb{Z}$.

- If $l_c - l_a = nl_{ca}$ for some integer $l_{ca}$, then we consider the function

  $$v_{ac}(s) = h(e^{-s\bar{F}}A_a, w_c).$$

  By Lemma 6.5.1, it is holomorphic when $|s| < 2$. Then

  – if $l_{ca} > 0$: when $|s| = 1$, by (6.15) $s^{l_{ca}} v_{ac}(s) = v_{ac}(1)$, which implies $v_{ac}(s) =$

$s^{-l_{ca}}v_{ac}(1)$ when $|s|=1$. However, by the properties of holomorphic functions, we must have $v_{ac}(s)=s^{-l_{ca}}v_{ac}(1)$ when $0<|s|<2$. Since $v_{ac}$ is a holomorphic function on $|s|<2$, we must have $v_{ac}(1)=h(e^{-\bar{F}}A_a,w_c)=0$.

$$h_k^s=0,\quad l(A_s)=l(A_j).$$

– if $l_{ca}\leq 0$: follows from the same argument as above, one has $v_{ac}(s)=s^{-l_{ca}}v_{ac}(1)$ when $|s|<2$. In particular, one must have

$$v_{ac}(1)=\frac{1}{(-l_{ca})!}(\frac{d}{ds})^{-l_{ca}}|_{s=0}v_{ac}(s),$$

i.e. $h(e^{-\bar{F}}A_a,w_c)=\frac{1}{(-l_{ca})!}\int_{\mathbb{C}^n}\bar{F}^{-l_{ca}}A_a\wedge *\bar{w}_c$.

In particular, when $l_{ca}=0$,

$$h(e^{-\bar{F}}A_a,w_c)=h(A_a,w_c)=h(w_a,w_c)-h(\bar{\partial}_{F(\cdot,u)}\nu_a,w_c)=h(w_a,w_c).$$

As a result, Theorem 1.3.9 follows.

□

**Lemma 6.5.1.** *The function* $v_{ac}(s)=\int_{\mathbb{C}^n}e^{-s\bar{F}}A_a\wedge *\bar{w}_c$ *is holomorphic on the disk* $|s|<2$.

*Proof.* This is because $[\bar{\partial}_{F(\cdot,u)},\bar{\partial}^*_{F(\cdot,u)}]=\frac{1}{2}[d_{2\mathrm{Re}(F)},d^*_{2\mathrm{Re}(F)}]$, and by Agmon estimate, there exists $C_b>0$ for any $b\in(0,1)$, s.t.

$$|w_c|\leq C_b e^{-b\rho}\|w_c\|_{L^2},$$

where $\rho(z)$ is the Agmon distance between $z$ and 0 with respect to Agmon metric $2|\nabla\mathrm{Re}(F)|^2$. It follows from the properties of Agmon distance and holomorphic functions that $\rho(z) \geq 2|\mathrm{Re}(e^{i\theta}F)|$ for any $\theta \in \mathbb{R}$. As a result, for any $a < b$, if $|s| \leq 2a$,

$$\int_{\mathbb{C}^n} e^{-s\bar{F}} A_a \wedge *\bar{w}_c \leq C \int_{\mathbb{C}^n} e^{2a\mathrm{Re}(F)} e^{-b\rho} \mathrm{dvol}_{\mathbb{C}^n} \leq C \int_{\mathbb{C}^n} e^{-(b-a)\rho} \mathrm{dvol}_{\mathbb{C}^n} < \infty.$$

Hence $v_{ac}(s)$ is holomorphic when $|s| < 2$. $\square$

## 6.6 LG/CY Correspondence for Period Integral

By Theorem 1.3.9, we also have

$$\int_{\gamma_k} e^{F+\bar{F}} w_a = \int_{\gamma_k} \left( e^F A_a - \sum_{b: l_b < l_a} (\mathcal{T})_{ab} e^F A_b \right). \tag{6.16}$$

**Definition 6.6.1.** *Now we define the map $r : \mathcal{H}' \to H^{n-2}(X_u)_0$ via*

$$r(w_a) = 2\pi i \left( (-1)^{n_a - 1}(n_a - 1)! R(\phi_a) - \sum_{b: l_b < l_a} (\mathcal{T})_{ab} (-1)^{n_b - 1}(n_b - 1)! R(\phi_b) \right),$$

*where $X_u := \{[z] \in \mathbb{C}P^{n-1} : F(z, u) = 0\}$.*

**Remark 6.6.2.** *Then by (6.12), one can see that if $\hat{p}(w) = p \in \mathbb{Z}$, then $r(w) \in F^p H^{n-2}(X_u)_0$.*

**Lemma 6.6.3.** *For $0 \leq k \leq \mu' - 1, \phi_a \in \mathrm{Jac}(F)'$,*

$$\int_{\gamma_k} e^F A_a = 2\pi i (-1)^{n_a - 1}(n_a - 1)! \int_{\delta_k} R(\phi_a),$$

*and*

$$\int_{\tilde{\gamma}_k} e^{-F} * A_a = 2\pi i (-1)^{n_a}(n_a - 1)! \int_{\delta_k} *R(\phi_a).$$

*Here $n_a = \frac{l_a}{n}$.*

*Proof.* It suffices to compute at $u = 0$.

Let $\sigma_k(t) = (\Phi_T)^*\sigma_k \in H_n(V_{-t})$. For each $t \neq 0$, let $\tilde{c}_t(\theta) := -t + \frac{te^{i\theta}}{2}$, $T(\sigma_k)(t) := \cup_\theta P_\theta(\sigma_k(-\frac{t}{2}))$, where $P_\theta : H_n(V_{-\frac{t}{2}}) \to H_n(V_{\tilde{c}_t(\theta)})$ is the parallel transport along $\tilde{c}_t$. Then

$$\begin{aligned}\int_{\gamma_k} e^f A_a &= \int_0^\infty e^{-t}\int_{\sigma_k(t)} \frac{A_a}{df}dt \\ &= \frac{1}{2\pi i}\int_0^\infty e^{-t}\int_{T(\sigma_k(t))} \frac{A_a}{f+t}dt.\end{aligned} \tag{6.17}$$

Let $v(t) = \int_{T(\sigma_k(t))} \frac{A_a}{f+t}$, then $v(t) = t^{n_a-1}v(1)$ (recall that $n_a = \frac{l_a}{n}$), proceed as in the proof of Lemma A.2 in [17], differential $t$, one has

$$v(t) = (-1)^{n_a-1}t^{n_a-1}\int_{T(\sigma_k(t))} \frac{A_a}{(f+t)^{n_a}}$$

It follows from Theorem 4.2 in [23] that

$$\int_{T(\sigma_k(t))} \frac{A_a}{(f+t)^{n_a}} = \lim_{t\to 0}\int_{T(\sigma_k(t))} \frac{A_a}{(f+t)^{n_a}} = (2\pi i)^2\int_{\delta_k} R(\phi_a).$$

□

**Proposition 6.6.4.** *For $0 \leq k \leq \mu' - 1$, one has*

$$\int_{\gamma_k} e^{F+\bar{F}}w = \int_{\delta_k} r(w) \tag{6.18}$$

*and*

$$\int_{\tilde{\gamma}_k} e^{-F-\bar{F}} * w = \int_{\delta_k} *r(w) \tag{6.19}$$

*for any $w \in \mathcal{H}'$.*

*Moreover, let $\hat{p}_a = \hat{p}(w_a), \hat{q}_a = \hat{q}(w_a)$, then $r(w_a) \in H^{\hat{p}_a,\hat{q}_a}(X_u)_0$ and $r(\bar{w}_a) = \overline{r(w_a)}$*

*for any* $\phi_a \in \mathrm{Jac}(f)'$*. As a result, by (6.12) and the definition of* $r$ *and* $R$*,*

$$r(w_a) = (2\pi i)(-1)^{n_a-1}(n_a-1)!R(\phi_a)^{\hat{p}_a,\hat{q}_a}, \tag{6.20}$$

*where for* $\alpha \in \mathcal{A}^k(X_u)$*,* $\alpha^{p,q}$ *denotes the* $p, q$ *part of* $\alpha$

*Proof.* By Lemma 6.6.3 and (6.16), for $0 \leq k \leq \mu' - 1$, $\forall w \in \mathcal{H}'$,

$$\int_{\gamma_k} e^{F+\bar{F}} w = \int_{\delta_k} r(w). \tag{6.21}$$

As a result,

$$\int_{\gamma_k} e^{F+\bar{F}} \bar{w}_a = \int_{\delta_k} r(\bar{w}_a).$$

Moreover,

$$\int_{\gamma_k} e^{F+\bar{F}} \bar{w}_a = \overline{\int_{\gamma_k} e^{F+\bar{F}} w_a} = \overline{\int_{\delta_k} r(w_a)} = \int_{\delta_k} \overline{r(w_a)}.$$

Thus,

$$r(\bar{w}_a) = \overline{r(w_a)} \in \overline{F^{\hat{p}_a} H^{n-2}(X_u)_0}, \tag{6.22}$$

One the other hand, since $\bar{w}_a = S(\kappa(\phi)_a dz_1...dz_n)$, by Remark 6.6.2 and Proposition 6.8, $r(\bar{w}_a) \in F^{\hat{q}_a} H^{n-2}(X_u)_0 = F^{n-\hat{p}_a} H^{n-2}(X_u)_0$. As a result,

$$r(\bar{w}_a) \in \overline{F^{\hat{p}_a} H^{n-2}(X_u)_0} \cap F^{n-\hat{p}_a} H^{n-2}(X_u)_0 = H^{\hat{p}_a,\hat{q}_a}(X_u)_0. \tag{6.23}$$

(6.22) and (6.23) tell us that $r(w_a) \in H^{\hat{p}_a,\hat{q}_a}(X_u)_0$. □

**Theorem 6.6.1.** *If* $\phi_a, \phi_b \in \mathrm{Jac}(F)'$*, then* $\int_{\mathbb{C}^n} w_a \wedge * w_b = \frac{(2\pi)^2}{2^{n-2}} \int_{X_u} r(w_a) \wedge * r(w_b)$*.*

*Proof.* By Lemma 6.1.1 and Lemma 6.2.1, if $l_a + l_b \neq n^2$, $\int_{\mathbb{C}^n} w_a \wedge * w_b = 0$. While by Proposition 6.6.4, if $l_a + l_b \neq n^2$, $\int_{X_u} r(w_a) \wedge * r(w_b) = 0$ too.

By (6.20), Theorem 3.4 in [53] and Theorem 3 in [21],

$$\begin{aligned}\int_{\mathbb{C}^n} w_a \wedge *w_b &= \frac{(-1)^{(n_a-1)(n_b-1)+1}(n_a-1)!(n_b-1)!}{2^{n-2}} \int_{X_u} R(\phi_a)^{\hat{p}_a,\hat{q}_a} \wedge *R(\phi_b)^{\hat{p}_b,\hat{q}_b} \\ &= \frac{(2\pi)^2}{2^{n-2}} \int_{X_u} r(w_a) \wedge *r(w_b).\end{aligned}$$

□

## 6.7 LG/CY Correspondence for Intersection Matrices

Let $(I)_{ij}(0 \leq i, j \leq \mu - 1)$ be the intersection matrix (see Definition 6.4.1), $\mathcal{I} = I^{-1}$; $I'_{ij}(0 \leq i, j \leq \mu' - 1)$ be a submatrix of $I$, and $\mathcal{I}' = (I')^{-1}$; $(I^{CY})_{ij} = \delta_i \cap \delta_j$ be the intersection matrix, and $\mathcal{I}^{CY} = (I^{CY})^{-1}$.

If $\phi_a \in \mathrm{Jac}(f)'$,

$$\int_{\gamma_k} e^F A_a = \int_0^\infty e^{-t} \int_{\sigma_k} \frac{A_a}{dF},$$

where $\frac{A_a}{dF}$ is the Gelfand-Leray form (c.f. Lemma 10.3 in [54]).

Integration by substitution tells us that

$$\int_{\sigma_k} \frac{A_a}{dF} = 0$$

if $M\sigma_k \neq \sigma_k$, which implies if $k \geq \mu'$

$$\int_{\gamma_k} e^F A_a = 0. \tag{6.24}$$

**Theorem 6.7.1.** $\mathcal{I}' = \frac{\pi^2}{2^{n-4}} \mathcal{I}^{CY}$.

*Proof.* On the one hand, by (6.24) and Riemann bilinear formula,

$$\begin{aligned}\int_{\mathbb{C}^n} w_a \wedge *w_b &= \left(\int_{\gamma_k} e^{F+\bar{F}} w_a\right) (\mathcal{I}')_{kl} \left(\int_{\tilde{\gamma}_l} e^{-F-\bar{F}} * w_b\right)\\ &= \left(\int_{\delta_k} r(w_a)\right) (\mathcal{I}')_{kl} \left(\int_{\delta_l} *r(w_b)\right) \text{ (By (6.16) and Lemma 6.6.3).}\end{aligned}$$

On the other hand, by Theorem 6.6.1

$$\begin{aligned}\int_{\mathbb{C}^n} w_a \wedge *w_b &= \frac{\pi^2}{2^{n-2}} \int_{X_u} r(w_a) \wedge *r(w_b)\\ &= \frac{\pi^2}{2^{n-2}} \left(\int_{\delta_k} r(w_a)\right) (\mathcal{I}^{CY})_{kl} \left(\int_{\delta_l} *r(w_b)\right).\end{aligned}$$

Hence, $\mathcal{I}' = \frac{\pi^2}{2^{n-2}} \mathcal{I}^{CY}$. ∎

*Proof of Theorem 1.3.10.* By Theorem 6.7.1 and Lemma 6.6.3, it's easy to see that

$$\int_{\mathbb{C}^n} w_0 \wedge *\bar{w}_0 = \frac{\pi^2}{2^{n-2}} \int_{X_u} r(w_0) \wedge *\overline{r(w_0)}.$$

Moreover, noticing that $G_{i\bar{j}} = \partial_i \bar{\partial}_{\bar{j}} \log(\int_{\mathbb{C}^n} w_0 \wedge *\bar{w}_0)$, $G^{CY}_{i\bar{j}} = \partial_i \bar{\partial}_{\bar{j}} \log(\int_{X_u} r(w_0) \wedge *\overline{r(w_0)})$, we have Theorem 1.3.10. ∎

To show $LG/CY$ correspondence for $tt^*$ geometry, we first show that:

**Lemma 6.7.1.**

$$\partial_i \int_{\gamma_k} e^{F+\bar{F}} w_a = \int_{\gamma_k} e^{F+\bar{F}} (D_i + C_i) w_a,$$

$$\bar{\partial}_{\bar{i}} \int_{\gamma_k} e^{F+\bar{F}} w_a = \int_{\gamma_k} e^{F+\bar{F}} (\bar{D}_{\bar{i}} + \bar{C}_{\bar{i}}) w_a.$$

*Proof.* First, notice that

$$\partial_i \int_{\gamma_k} e^{F+\bar{F}} w_a = \int_{\gamma_k} \partial_i (e^{F+\bar{F}} w_a) = \int_{\gamma_k} e^{F+\bar{F}} (\partial_i + \phi_i) w_a. \tag{6.25}$$

Since $de^{F+\bar{F}} w_a$ is $d$-closed for all $u$, $e^{F+\bar{F}}(\partial_i + \phi_i) w_a$ is also $d$-closed, which implies that $(\partial_i + \phi_i) w_a$ is $d_{2\mathrm{Re}(F)}$-closed. Proceeding as in the proof of Proposition 6.4.4, one shows that

$$(\partial_i + \phi_i) w_a = \Pi_u((\partial_i + \phi_i) w_a) + d_{2\mathrm{Re}(F)} \alpha = (D_i + C_i) w_a + d_{2\mathrm{Re}(F)} \alpha$$

for some differential form $\alpha$ with exponential decay.

As a result,

$$\int_{\gamma_k} e^{F+\bar{F}} (\partial_i + \phi_i) w_a = \int_{\gamma_k} e^{F+\bar{F}} (D_i + C_i) w_a. \tag{6.26}$$

By (6.25) and (6.26), one has

$$\partial_i \int_{\gamma_k} e^{F+\bar{F}} w_a = \int_{\gamma_k} e^{F+\bar{F}} (D_i + C_i) w_a.$$

Similarly, one can show that

$$\bar{\partial}_{\bar{i}} \int_{\gamma_k} e^{F+\bar{F}} w_a = \int_{\gamma_k} e^{F+\bar{F}} (\bar{D}_{\bar{i}} + \bar{C}_{\bar{i}}) w_a.$$

□

*Proof of Theorem 1.3.11.* First, let $r' = 2^{\frac{4-n}{2}} \pi r$. By Theorem 6.6.1 and Proposition 6.6.4, it suffices to prove

$$\begin{aligned} r'(Dw_a) = D^{CY} r'(w_a), r'(\bar{D}w_a) = \bar{D}^{CY} r'(w_a) \\ r'(Cw_a) = C^{CY} r'(w_a), r'(\bar{C}w_a) = \bar{C}^{CY} r'(w_a) \end{aligned} \tag{6.27}$$

for $\phi_a \in \mathrm{Jac}(f)'$.

Then $r' : \mathcal{H}' \to H_{n-2}(X_u)_0$ induced an isomorphism of $tt^*$ structure.

By (6.18), one has

$$\partial_i \int_{\gamma_k} e^{F+\bar{F}} w = \partial_i \int_{\delta_k} r'(w),$$
$$\bar{\partial}_{\bar{i}} \int_{\gamma_k} e^{F+\bar{F}} w = \bar{\partial}_{\bar{i}} \int_{\delta_k} r'(w).$$

Hence by Lemma 6.7.1,

$$r'\left((D+C)w_a\right) = (D^{CY} + C^{CY}) r'(w_a), r'\left((\bar{D}+\bar{C})w_a\right) = (\bar{D}^{CY} + \bar{C}^{CY}) r'(w_a). \tag{6.28}$$

Suppose $w = A = \phi dz_1 ... dz_n$ such that $l(A) = l(A_a)$, then By Riemann bilinear formula and Proposition 6.6.4,

$$\int_{\mathbb{C}^n} w_a \wedge * \bar{w} = \int_{X_u} r'(w_a) \wedge * \overline{r'(w)}.$$

As a result,

$$\partial_i \int_{\mathbb{C}^n} w_a \wedge * \bar{w} = \partial_i \int_{X_u} r'(w_a) \wedge * \overline{r'(w)}.$$

Then noticing that

$$\partial_i \int_{\mathbb{C}^n} w_a \wedge * \bar{w} = \int_{\mathbb{C}^n} D_i w_a \wedge * \bar{w},$$

and

$$\partial_i \int_{X_u} r'(w_a) \wedge * \overline{r(w)} = \int_{X_u} D_i^{CY} r'(w_a) \wedge * \overline{r'(w)},$$

one has,

$$r'(Dw_a) = D^{CY} r'(w_a).$$

Similarly,

$$r(\bar{D}w_a) = \bar{D}^{CY} r(w_a).$$

Together with (6.28), we have (6.27).

□

# Appendix A

# Kodaira-Hodge Decomposition

In this section, we investigate the decomposition (4.1). For this purpose we first have to understand the Friedrichs extension of $\Delta_{H,f}$. Here we assume that all operators considered in this section are closable, as are our $d_{Tf}, \delta_{Tf}$ (Cf. [55, Theorem VIII.1]).

## A.1 Review on Friedrichs Extension

Let A be a nonnegative, symmetric (unbounded) operator on Hilbert space $\mathcal{H}$, with $\mathrm{Dom}(\mathrm{A}) = V$, i.e.

$$(\mathrm{A}\,\alpha, \beta)_{\mathcal{H}} = (\alpha, \mathrm{A}\,\beta)_{\mathcal{H}},\ \forall \alpha, \beta \in V;\quad (\mathrm{A}\,\alpha, \alpha)_{\mathcal{H}} \geq 0.$$

Define a norm $\|\cdot\|_{V_1}$ on $V$ by

$$\|\alpha\|_{V_1}^2 = (\alpha, \alpha)_{\mathcal{H}} + (\alpha, \mathrm{A}\,\alpha)_{\mathcal{H}}.$$

Let $V_1$ to be the completion of $V$ under $\|\cdot\|_{V_1}$. Then for any $\beta \in \mathcal{H}$, one can construct a bounded linear functional $L_\beta$ on $V_1$ as follows

$$L_\beta(\phi) = (\phi, \beta)_{\mathcal{H}}, \phi \in V_1. \tag{A.1}$$

Since $|(\phi, \beta)_{\mathcal{H}}| \leq \|\phi\|_{\mathcal{H}} \|\beta\|_{\mathcal{H}} \leq \|\phi\|_{\mathcal{H}} \|\beta\|_{V_1}$, $L_\beta$ is indeed bounded functional on $V_1$. By Riesz representation, there exist $\gamma \in V_1$, s.t. $(\phi, \gamma)_{V_1} = (\phi, \beta)_{\mathcal{H}}$.

Let $\mathrm{B} : \mathcal{H} \to V_1, \beta \mapsto \gamma$, then B is bounded and injective. Taking $\square = \mathrm{B}^{-1} - \mathrm{I}$, where I is the identity (inclusion) map, then $\square$ is the Friedrichs extension of A, with $\mathrm{Dom}(\square) = \mathrm{Im}(\mathrm{B})$.

**Remark A.1.1.** *From the construction of Friedrichs extension* $\square$ *of* A*, we can see that* *Dom*$(\square) = \mathrm{Im}((\mathrm{I} + \square)^{-1})$.

Let $\mathrm{T}, \mathrm{S}$ be two unbounded operators on Hilbert space $\mathcal{H}$, s.t.

(i)

$$V = \mathrm{Dom}(\mathrm{T}) = \mathrm{Dom}(\mathrm{S}), \mathrm{T}\, V \subset V.$$

(ii) $S$ is a formal adjoint of $T : \forall \alpha, \beta \in V$,

$$(\mathrm{T}\,\alpha, \beta)_{\mathcal{H}} = (\alpha, \mathrm{S}\,\beta)_{\mathcal{H}},$$

Let $\|\cdot\|_W$ be the norm on $V$ given by

$$\|\alpha\|_W^2 = (\alpha, \alpha)_{\mathcal{H}} + (\mathrm{T}\,\alpha, \mathrm{T}\,\alpha)_{\mathcal{H}}, \alpha \in V,$$

and $W$ be the completion of $V$ under the norm $\|\cdot\|_W$. Then we can extend T to $\bar{\mathrm{T}}_{min}$ with $\mathrm{Dom}(\bar{\mathrm{T}}_{min}) = W$.

Let $\bar{\mathrm{S}}_{max}$ be the closure of S with $\mathrm{Dom}(\bar{\mathrm{S}}_{max}) = \{\alpha \in \mathcal{H} : |(\alpha, \mathrm{T}\,\phi)_{\mathcal{H}}| \leq M_\alpha \|\phi\|_{\mathcal{H}}, \forall \phi \in V\}$. Namely, for any $\alpha \in \mathrm{Dom}(\bar{\mathrm{S}}_{max})$, since $V$ is dense in $\mathcal{H}$, by Riesz representation, there exists a unique $\nu \in \mathcal{H}$, such that $(\nu, \phi)_H = (\alpha, \mathrm{T}\,\phi)$. Now define $\bar{\mathrm{S}}_{max}(\alpha) = \nu$.

Since $\mathrm{T}\,V \subset V$, $\mathrm{S}\,\mathrm{T}$ is symmetric and nonnegative with $\mathrm{Dom}(ST) = V$.

**Proposition A.1.2.** *The Friedrichs extension* $\Delta$ *of* $\mathrm{S}\,\mathrm{T}$ *is just* $\bar{\mathrm{S}}_{max}\bar{\mathrm{T}}_{min}$.

*Proof.* Since $\mathrm{T}\,V \subset V$, we see that $V_1$ constructed in (A.1) is the same as $W$. Indeed, for any $\phi, \psi \in V$, we have

$$(\psi, \phi)_{\mathcal{H}} + (\mathrm{T}\,\psi, \mathrm{T}\,\phi)_{\mathcal{H}} = (\psi, \phi)_{\mathcal{H}} + (\mathrm{S}\,\mathrm{T}\,\psi, \phi)_{\mathcal{H}}$$

Hence, we have

$$\mathrm{Dom}(\Delta) = \{\alpha \in W : \alpha = (I + \Delta)^{-1} f, f \in \mathcal{H}\},$$

$$\mathrm{Dom}(\bar{S}_{max}\bar{T}_{min}) = \{\alpha \in W : \bar{\mathrm{T}}_{min}\alpha \in \mathrm{Dom}(\bar{\mathrm{S}}_{max})\}.$$

We now divide our discussion in two cases.

(a) We first prove that $\mathrm{Dom}\bar{\mathrm{S}}_{max}\bar{\mathrm{T}}_{min} \subset \mathrm{Dom}(\Delta)$, and $\forall \alpha \in \mathrm{Dom}(\bar{\mathrm{S}}_{max})$, $\bar{\mathrm{S}}_{max}\bar{\mathrm{T}}_{min}\alpha = \Delta\alpha$.

For any $\alpha \in \mathrm{Dom}\bar{\mathrm{S}}_{max}\bar{\mathrm{T}}_{min}$, let

$$\beta = \alpha + \bar{\mathrm{S}}_{max}\bar{\mathrm{T}}_{min}\alpha. \tag{A.2}$$

Then for any $\phi \in W$, we have

$$\begin{aligned}
(\alpha, \phi)_W &= \lim_{n\to\infty} (\alpha, \phi_n)_W \\
&= \lim_{n\to\infty} (\alpha, \phi_n)_{\mathcal{H}} + (\bar{\mathrm{T}}_{min}\alpha, \mathrm{T}\,\phi_n)_{\mathcal{H}} \\
&= \lim_{n\to\infty} (\alpha, \phi_n)_{\mathcal{H}} + (\bar{\mathrm{S}}_{max}\bar{\mathrm{T}}_{min}\alpha, \phi_n)_{\mathcal{H}} \text{ (Since } \phi_n \in V, \bar{\mathrm{T}}_{min}\alpha \in \mathrm{Dom}(\bar{\mathrm{S}}_{max})\text{ )} \\
&= \lim_{n\to\infty} (\alpha + \bar{\mathrm{S}}_{max}\bar{\mathrm{T}}_{min}\alpha, \phi_n)_{\mathcal{H}} = (\alpha + \bar{\mathrm{S}}_{max}\bar{\mathrm{T}}_{min}\alpha, \phi)_{\mathcal{H}} \\
&= (\beta, \phi)_{\mathcal{H}},
\end{aligned} \tag{A.3}$$

where $\phi_n \in V$, and $\phi_n \to \phi$ in $\|\cdot\|_W$ norm. By the construction of Friedrichs extension and (A.3), we deduce that $\alpha \in (I+\Delta)^{-1}\mathcal{H}$ and $(I+\Delta)\alpha = \beta$. Comparing with (A.2), we obtain $\bar{\mathrm{S}}_{max}\bar{\mathrm{T}}_{min}\alpha = \Delta\alpha$.

(b) We then show that $\mathrm{Dom}(\Delta) \subset \mathrm{Dom}(\bar{\mathrm{S}}_{max}\bar{\mathrm{T}}_{min})$.

Take any $\alpha \in \mathrm{Dom}(\Delta) \subset W$, we can find $f \in \mathcal{H}$, s.t. $\alpha = (I+\Delta)^{-1}f$. We now just need to show that $\bar{\mathrm{T}}_{min}\alpha \in \mathrm{Dom}(\bar{\mathrm{S}}_{max})$. For this, it suffices to prove that $\forall g \in V$, $|(\bar{\mathrm{T}}_{min}\alpha, \mathrm{T}\,g)_{\mathcal{H}}| \leq M\|g\|_{\mathcal{H}}$ for some $M > 0$.

In fact, by standard functional calculus,

$$\begin{aligned}
&|(\bar{\mathrm{T}}_{min}\alpha, \mathrm{T}\,g)_{\mathcal{H}}| = |(\alpha, STg)_{\mathcal{H}}| \text{ (via } \alpha_n \in V, \alpha_n \to \alpha \text{ in } \|\|_W) \\
&= |((I+\Delta)^{-1}f, \Delta g)_{\mathcal{H}}| \\
&= |(f, (I+\Delta)^{-1}\Delta g)_{\mathcal{H}}| \\
&\leq M\|g\|_{\mathcal{H}}
\end{aligned}$$

□

## A.2 The Friedrichs Extension of $\Delta_{H,f}$

By Proposition A.1.2, the Friedichs extension $\Box_f$ of $\Delta_{H,f}$ is

$$(\overline{d_f+\delta_f})_{max}(\overline{d_f+\delta_f})_{min}.$$

If 0 is an eigenvalue of $\Box_f$ with finite multiplicity, we have the following decomposition

$$L^2\Lambda^*(M) = \ker\Box_f \oplus \mathrm{Im}(\overline{d_f+\delta_f})_{max}. \tag{A.4}$$

Could we say more about decomposition (A.4)?

**Proposition A.2.1.** *Let* $\mathrm{T}, \mathrm{S}$ *be two unbounded operators on Hilbert space* $\mathcal{H}$*, such that*

1. $$V = \mathrm{Dom}(\mathrm{T}) = \mathrm{Dom}(\mathrm{S}), \mathrm{T}\,V \subset V.$$

2. $\mathrm{Im}(T)$ *is orthogonal to* $\mathrm{Im}(S)$*, and*

$$(\mathrm{T}\,\alpha, \beta)_{\mathcal{H}} = (\alpha, \mathrm{S}\,\beta)_{\mathcal{H}}.$$

3. $T+S$ *is essential self-adjoint, i.e.* $\overline{(T+S)}_{min} = \overline{(T+S)}_{max}$.

*Then*

$$\overline{T+S} = \bar{T}_{min}|_{\mathrm{Dom}\bar{S}_{min}\cap\mathrm{Dom}\bar{T}_{min}} + \bar{S}_{min}|_{\mathrm{Dom}\bar{S}_{min}\cap\mathrm{Dom}\bar{T}_{min}}$$

$$= \bar{T}_{max}|_{\mathrm{Dom}\bar{S}_{max}\cap\mathrm{Dom}\bar{T}_{max}} + \bar{S}_{max}|_{\mathrm{Dom}\bar{S}_{max}\cap\mathrm{Dom}\bar{T}_{max}}$$

*Proof.* Since $\mathrm{Dom}\overline{(T+S)}_{min}$ is the closure of $V$ under metric

$$(\phi,\phi)_{\mathcal{H}} + ((T+S)\phi, (T+S)\phi)_{\mathcal{H}} = (\phi,\phi)_{\mathcal{H}} + (T\phi, T\phi)_{\mathcal{H}} + (S\phi, S\phi)_{\mathcal{H}}, (**)$$

we have $\mathrm{Dom}\overline{(T+S)}_{min} \subset \mathrm{Dom}\bar{S}_{min} \cap \mathrm{Dom}\bar{T}_{min}$. Also, for any $\phi \in \mathrm{Dom}(T+S)_{min}$

$$\overline{(T+S)}_{min}\phi = \lim_{n\to\infty}(T+S)\phi_n = \lim_{n\to\infty} T\phi_n + S\phi_n = \mathrm{T}_{min}\,\phi + \mathrm{S}_{min}\,\phi,$$

where $\phi_n \in V \to \phi$ in the metric $(**)$.

For each $\phi \in \mathrm{Dom}\bar{S}_{max} \cap \mathrm{Dom}\bar{T}_{max}$, $\psi \in V$,

$$\begin{aligned}
&(\phi,(T+S)\psi)_{\mathcal{H}} = (\phi, T\psi)_{\mathcal{H}} + (\phi, S\psi)_{\mathcal{H}} \\
&= (\bar{\mathrm{T}}_{max}\phi, \psi)_{\mathcal{H}} + (\bar{\mathrm{S}}_{max}\phi, \psi)_{\mathcal{H}} \\
&\le C\|\psi\|_{\mathcal{H}}.
\end{aligned}$$

Therefore $\phi \in \mathrm{Dom}(\overline{(T+S)}_{max})$, and $\overline{(T+S)}_{max}\phi = \bar{\mathrm{T}}_{max}\phi + \bar{\mathrm{S}}_{max}\phi$, which means that $\mathrm{Dom}\bar{S}_{min} \cap \mathrm{Dom}\bar{T}_{min} \subset \mathrm{Dom}(\overline{(T+S)}_{max})$. $\square$

Our Theorem 3.1.3, the Kodaira decomposition for the Witten decomposition, follows from (A.4), Theorem 3.1.1 and Proposition A.2.1 above.

# Appendix B

# More Refined Agmon Estimate

This chapter establishes a more precise Agmon estimate, which will be needed in Chapter 6. Following that, we move on to prove Proposition 6.4.4.

## B.1 Agmon estimate

In this section, we let $V \in C^\infty(\mathbb{R}^n)$ be a nonnegative function with finite isolated zeros. Moreover

$$\lim_{|x|\to\infty} \frac{|\nabla V|}{(V+1)^{3/2}}(x) = 0.$$

The metric $\tilde{g} := Vg_0$ is called Agmon metric with respect to $V$, where $g_0$ is the standard metric on $\mathbb{R}^n$. Let $\tilde{d}$ is the distance function induced by the Agmon metric, and $\rho(x) := \tilde{d}(x, 0)$. We summarize several nice properties of Agmon distance here (c.f. [25] )

**Lemma B.1.1.** *1. $|\nabla\rho|^2 = V$, a.e.*

*2. If $|\nabla f| \leq V$, then $|f(x) - f(y)| \leq \tilde{d}(x, y)$. In particular, if $f(0) = 0$, $|f|(x) \leq \rho(x)$.*

**Lemma B.1.2.** *Assume that $w \in L^2(\mathbb{R}^n), 0 \leq u \in L^2(M)$, s.t. $(\Delta + V)u \leq w$ outside a compact subset $K \subset M$ in the weak sense (where the interior of $K$ contains all the zeros* *of* $V$*). That is*

$$\int_{\mathbb{R}^n - K} \nabla u \nabla v + V uv \mathrm{dvol} \leq \int_{\mathbb{R}^n - K} w \cdot v \mathrm{dvol}, \quad \forall\ 0 \leq v \in C_c^\infty(M - K).$$

*If* $\int_{\mathbb{R}^n - K} |w|^2 V^{-1} \exp(2b\rho) \mathrm{dvol} < \infty$ *for some* $b \in (0,1)$*, then for any compact set such that* $L^\circ \supset K$*, one has*

$$\begin{aligned} \int_{\mathbb{R}^n - L} V|u|^2 \exp(2b\psi) \mathrm{dvol} \leq C(b, K, L) \int_{\mathbb{R}^n - K} |w|^2 V^{-1} \exp(2b\psi) \mathrm{dvol} \\ + C(b, K, L) \int_{L - K} V|u|^2 \exp(2b\psi) \mathrm{dvol}. \end{aligned} \tag{B.1}$$

*Proof.* This is exactly Lemma 3.1 in [25]. □

First, let's recall the De Giorgi-Nash-Moser theorem:

**Theorem B.1.1.** *Let* $B_r := \{x \in \mathbb{R}^n : |x| < r\}$*. Suppose that* $u \in L^2(B_r)$*,* $w \in L^N(B_r)$ *for some* $N > n/2$*, s.t.* $\Delta u \leq w$ *in the weak sense (and* $u \geq 0$*.). Then*

$$\sup_{y \in B_r} u(y) \leq C(r^{-n/2} \|u\|_{L^2(B_{2r})} + r^{-n/N} \|w\|_{L^N(B_{2r})}),$$

*where* $C > 0$ *is a constant depending on* $n$ *and* $N$*.*

*Proof.* See Theorem 4.1 in [41] for a reference. □

**Lemma B.1.3.** *Suppose* $u, w$ *satisfy the same conditions as in Lemma B.1.2 for any compact set containing the zeros of* $V$*. Moreover, assume that* $w$ *and satisfies*

$$\|w\|_{L^N_{wt}}^N := \int_M |w|^N \exp(Nb\rho) dvol < \infty,$$

$$\|w\|_{L^2_{wt}}^2 := \int_M |w|^2 \exp(2b\rho) dvol < \infty,$$

*where* $N^* = \frac{N}{N-1}$. *Then for* $a \in (0,b)$

$$|u| \leq C(\psi, V, n, N, a, b)\left(\|u\|_{L^2} + \|w\|_{L^N_{wt}} + \|w\|_{L^2_{wt}}\right)\exp(-a\rho)$$

*Proof.* Fist, we fixed a compact subset $K$, such that outside $K$, $\frac{|\nabla V|}{(V+1)^{3/2}} \leq \frac{(b-a)}{2}$, and let $L := \{x \in \mathbb{R}^n : \tilde{d}(x,K) \leq 3\}$.

Denoted $\tilde{B}_r(x) := \{y \in \mathbb{R}^n : \tilde{d}(y,x) < r\}$. For $x_0 \notin L$, set $l = \sup_{x\in\tilde{B}_2(x_0)} V(x)$, and $r = 1/(2l)$. Then one can easily verify that $B_{2r}(x_0) \subset \tilde{B}_1(x_0)$.

Choose $y_0 \in \overline{\tilde{B}_2(x_0)}$ so that $V(y_0) \in (l/2, l]$. By Lemma B.1.2 and de Giorgi-Nash-Moser estimate (Theorem B.1.1),

$$\begin{aligned}
&|u(x_0)|^2 e^{2b\rho(x_0)} \leq C(n,N)(r^{-n}\|u\|^2_{L^2(B_{2r}(x_0))}e^{2b\rho(x_0)} + r^{-2n/N}\|w\|^2_{L^N(B_{2r}(x_0))}e^{2b\rho(x_0)})\\
&\leq C(n,N,b)\left(r^{-n}\int_{\tilde{B}_1(x_0)}|u|^2(y)e^{2b\rho(y)}dy + r^{-2n/N}(\int_{\tilde{B}_1(x_0)}|w(y)|^N e^{Nb\rho(y)}dy)^{2/N}\right)\\
&\leq C(n,N,b,a,V)\left(r^{-n}\int_{L-K}|u|^2e^{2b\rho}dy + r^{-n}\int_{\mathbb{R}^n-K}|w|^2e^{2b\rho}dy + r^{-2n/N}\|w\|^2_{L^N_{wt}}\right)\\
&\leq C(n,N,b,a,V)\left(|V(y_0)|^n\|u\|^2_{L^2} + |V(y_0)|^n\|w\|^2_{L^2_{wt}} + |V(y_0)|^{2n/N}\|w\|^2_{L^N_{wt}}\right)
\end{aligned}$$

Proceeding as in [25], one has

$$|V(y_0)|^2 \leq C(V,a,b)\exp((b-a)\rho(y_0)). \tag{B.2}$$

Hence, the result follows.

For $x_0 \in L$, we have classical de Giorgi-Nash-Moser estimate

$$|u|(x_0) \leq C(a,b,V,N,n)(\|u\|_{L^2} + \|w\|_{L^N}).$$

Since in $L$, $\exp(-a\rho)$ has an upper and a lower bound depending on $a, b$ and $V$, the result follows. □

## B.2 Witten deformation and Agmon estimate

Suppose that $f$ is a non-degenerate homogeneous polynomial on $\mathbb{C}^n$, then for any $a \in (0,1)$, there exists $r_0 := r_0(a) > 0$, s.t outside $B_{r_0} := \{x \in \mathbb{R}^n : |x| \leq r_0\}$, the Witten Laplacian $\Delta_{2\mathrm{Re}(f)} \geq \Delta + a|2\nabla \mathrm{Re}(f)|^2$ (c.f. [25] and [27]), i.e. for any smooth form $w$,

$$g_0(\Delta_{2\mathrm{Re}(f)} w, w)(p) \geq g_0((\Delta + a|2\nabla \mathrm{Re}(f)|^2)w, w)(p)$$

for any point $p \notin B_{r_0}$.

Then if $\Delta_{2\mathrm{Re}(f)} u = v$ for some differential forms $u$ and $v$, Bochner formula and Kato's inequality tells us that

$$(\Delta + a|2\nabla \mathrm{Re}(f)|^2)|u| \leq |v| \tag{B.3}$$

weakly.

In this case, let $\rho$ be the Agmon distance with respect to $V := |\nabla 2\mathrm{Re}(f)|^2$. The Agmon estimate discussed in the previous section is applicable for Witten deformation.

## B.3 Proof of Proposition 6.4.4

It suffices to prove the case of $u = 0$.

Recall that by our construction, $\gamma_k = \sup_{t>0} \Phi_t^*(\sigma_k)$ for some $\sigma_k \in H_{n-2}(V_{-1})$. Let

$$\epsilon_1 := (2 \sup_{z \in \cup_{k=0}^{\mu-1} \sigma_k} |z|)^{-n}.$$

Let

$$V := \{z \in \mathbb{C}^n : |z| = 1, |\mathrm{Re}(f)| \geq \epsilon_1\},$$

and $U$ be the cone

$$U := \{z \in \mathbb{C}^n : \frac{z}{|z|} \in V\},$$

then one has $\cup_{k=0}^{\mu-1} \gamma_k \subset U$.

**Lemma B.3.1.** *For every sufficiently small $\epsilon > 0$, there exists a smooth function $\tilde{f} : \mathbb{C}^n \to \mathbb{R}$, s.t.*

- $\tilde{f} \geq |\mathrm{Re}(f)|$. *Moreover,* $\tilde{f} = |\mathrm{Re}(f)|$ *in* $U$.
- $|\nabla \tilde{f}| \leq (1 + \epsilon c(n, f))|\nabla \mathrm{Re}(f)|$ *for some* $c(n, f) > 0$. *Hence by Proposition B.1.1, we also have*
$$\tilde{f} \leq \frac{(1 + \epsilon c(n, f))}{2} \rho. \tag{B.4}$$
- $\tilde{f} \geq \epsilon |z|^n$.

*Proof.* Let $\eta \in C^\infty(\mathbb{R})$, $\epsilon < \epsilon_1$, s.t.

- $\epsilon/2 \leq \eta(x) \leq \epsilon$ if $|x| \leq \epsilon$, and $\eta(x) = |x|$, if $|x| \geq \epsilon$;
- $\eta \geq \epsilon/2$;
- $|\eta'| \leq 1$.

Let $(r, \theta)$ be the polar coordinates of $\mathbb{C}^n$, and $\nabla^\theta$ be gradient with respect to the standard metric on $S^{2n-1} := \{z \in \mathbb{C}^n : |z| = 1\}$. Now set $\tilde{f}(r, \theta) := r^n \eta \circ \mathrm{Re}(f)(1, \theta)$.

By our construction, one can see that

-
$$\epsilon |z|^n/2 \leq \tilde{f} \leq \epsilon |z|^n \text{ if } z \notin U, \text{ and } \tilde{f}(z) = |\mathrm{Re}(f)|(z) \text{ if } z \in U \tag{B.5}$$

- $$\tilde{f} \geq \epsilon |z|^n/2; \tag{B.6}$$

- $$|\nabla^\theta \tilde{f}| \leq |\nabla^\theta \mathrm{Re}(f)|. \tag{B.7}$$

Note that in $U$, $|\nabla \tilde{f}| = |\nabla \mathrm{Re}(f)|$. In polar coordinates,

$$\begin{aligned} |\nabla \mathrm{Re}(f)|^2(r,\theta) &= n^2 r^{2n-2} |\mathrm{Re}(f)|^2(1,\theta) + r^{2n-2} |\nabla^\theta \mathrm{Re}(f)(1,\theta)|^2, \\ |\nabla \tilde{f}|^2(r,\theta) &= n^2 r^{2n-2} \tilde{f}(1,\theta) + r^{2n-2} |\nabla^\theta \tilde{f}(1,\theta)|^2. \end{aligned} \tag{B.8}$$

Let $\epsilon_2 := \inf_{|z|=1} |\nabla \mathrm{Re}(f)|$, then by (B.5), (B.7) and (B.8), one can see easily that outside $U$,

$$|\nabla \tilde{f}| \leq (1 + \frac{\epsilon}{n\epsilon_2}) |\nabla \mathrm{Re}(f)|.$$

□

Notice that $e^{-\bar{f}} A_a$ is $d_{2\mathrm{Re}(f)}$ closed, although $e^{-\bar{f}} A_a$ is not $L^2$ integrable, one still have the following formulation of Hodge decomposition:

**Lemma B.3.2.** *One has the following decomposition*

$$e^{-\bar{f}} A_a = w' + d_{2\mathrm{Re}(f)} \beta,$$

*where $w'$ is a harmonic form. Moreover, $\beta$ satisfies*

- *$e^{f+\bar{f}}\beta$ has exponential decay on $U$, i.e., there exists $c, C > 0$, s.t. $|e^{f+\bar{f}}\beta| \leq Ce^{-c|z|^n}$ in $U$.*

- *There exists $a \in (0,1), C > 0$, such that $|\beta| \leq Ce^{2a\rho}$, where $\rho$ is the Agmon distance.*

Once we have the decomposition in the lemma, one can see

$$\int_{\gamma_k} e^f A_a = \int_{\gamma_k} e^{f+\bar{f}} w'$$

and

$$\int_{\mathbb{C}^n} e^{-\bar{f}} A_a \wedge *w = \int_{\mathbb{C}^n} w' \wedge *w.$$

(This is because, since $\beta$ satisfied the estimate above, by Stoke formula

$$\int_{\gamma_k^-} de^{f+\bar{f}}\beta = 0,$$

and integration by parts,

$$\int_{\mathbb{C}^n} d_{2\mathrm{Re}(f)}\beta \wedge *w = \int_{\mathbb{C}^n} \beta \wedge *d^{\dagger}_{2\mathrm{Re}(f)} w = 0.)$$

By Proposition 6.4.3, one has Proposition 6.4.4.

Now it suffices to prove Lemma B.3.2.

*Proof of Lemma B.3.2.* We fix a function $\tilde{f}$ that satisfies the conditions in Lemma B.3.1 for a fixed $\epsilon < \min\{\epsilon_1, \frac{1}{16c(n,f)}\}$.

Step 1 Let $d_{tw,0} := e^{-2\mathrm{Re}(f)-3\tilde{f}/2} \circ d \circ e^{2\mathrm{Re}(f)+3\tilde{f}/2} = e^{-3\tilde{f}/2} \circ d_{2\mathrm{Re}(f)} \circ e^{3\tilde{f}/2}$, and $\Delta_{tw,0} := d_{tw,0}d^*_{tw,0} + d^*_{tw,0}d_{tw,0}$ be the Witten Laplacian with respect to $d_{tw,0}$. Then one can see that $e^{-3\tilde{f}/2-\bar{f}}A_a \in L^2(\mathbb{C}^n)$ (since $|e^{-3\tilde{f}/2-\bar{f}}A_a| \leq Ce^{-\tilde{f}/2}|z|^{l_a} \leq Ce^{-\epsilon|z|^n/2}|z|^{l_a}$), and

$$d_{tw}e^{-3\tilde{f}/2-\bar{f}}A_a = 0.$$

As a result, we have Hodge decomposition (c.f. [25])

$$e^{-3\tilde{f}/2-\bar{f}}A_a = w_0 + d_{tw,0}\beta_0,$$

where $w_0$ is $\Delta_{tw,0}$-harmonic, and we can also assume that $\beta_0 \in \mathrm{Im}(d^*_{tw,0})$.

Since $\left|\nabla\left(2\mathrm{Re}(f) + 3\tilde{f}/2\right)\right| \geq 2|\nabla \mathrm{Re}(f)| - 3/2|\nabla \tilde{f}| \geq 13/32|\nabla \mathrm{Re}(f)|$, Agmon estimate tells us that

$$|w_0| \leq Ce^{-3\rho/16} \tag{B.9}$$

for some $C > 0$.

Moreover, by our choice of $\beta_0$, one has

$$d^*_{tw,0}e^{-3\tilde{f}/2-\bar{f}}A_a = d^*_{tw,0}\tilde{w}_0 + d^*_{tw,0}d_{tw,0}\beta_0 = d^*_{tw,0}d_{tw,0}\beta_0 = \Delta_{tw,0}\beta_0.$$

Let $\epsilon_3 := n\sup_{|z|=1}|\nabla 2\mathrm{Re}(f)|$, then $\rho \leq \epsilon_3|z|^n$. Hence, there exists $b > 0$, s.t. $e^{b\rho}d^*_{tw,0}e^{-3\tilde{f}/2-\bar{f}}A_a$ is both $L^N$ and $L^2$ integrable for some $N > n$. Hence, by Agmon estimate again,

$$|\beta_0| \leq Ce^{-b\rho}. \tag{B.10}$$

Now let $\tilde{\beta}_0 := e^{3\tilde{f}/2}\beta_0$, $\tilde{w}_0 = e^{3\tilde{f}/2}w_0$, then $e^{-\bar{f}}A_a = \tilde{w}_0 + d_{2\mathrm{Re}(f)}\tilde{\beta}_0$. Moreover, $\tilde{w}_0$ is $d_{2\mathrm{Re}(f)}$-closed. In $U$, $\tilde{f} = |\mathrm{Re}(f)|$, hence $|e^{f+\bar{f}}\tilde{\beta}_0| \leq Ce^{-f/2-a\rho} \leq Ce^{-c|z|^n}$; and $|\tilde{\beta}_0| \leq Ce^{3\tilde{f}/2-b\rho} \leq Ce^{(3(1+\epsilon)/4-b)\rho}$. Hence $\tilde{\beta}_0$ satisfies the estimate stated above.

Step 2 However, although $\tilde{w}_0$ is $d_{2\mathrm{Re}(f)}$-closed, it is not harmonic. Also, $\tilde{w}_0$ may not be $L^2$ integrable. To continue, we will use techniques similar to those used in Section 7.5 of [25]. Let $d_{tw,1} = e^{-\tilde{f}/4} \circ d_{tw,0} \circ e^{\tilde{f}/4} = e^{-5/4\tilde{f}-2\mathrm{Re}(f)} \circ d \circ e^{5/4\tilde{f}+2\mathrm{Re}(f)}$, and $\Delta_{tw,1}$ be the Witten Laplacian with respect to $d_{tw,1}$.

By (B.9) and (B.4), $\alpha_1 := e^{\tilde{f}/4}w_0$ is $d_{tw,1}$ closed and $L^2$-integrable. Hence, we have Hodge decomposition

$$\alpha_1 = w_1 + d_{tw,1}\beta_1,$$

where $w_1$ is $\Delta_{tw,1}$ harmonic, and we may assume $\beta_1 \in \mathrm{Im}(d^*_{tw,1})$.

Since

$$\begin{aligned}\left|\nabla\left(5/4\tilde{f} + 2\mathrm{Re}(f)\right)\right| &\geq 2|\nabla \mathrm{Re}(f)| - 5/4|\nabla\tilde{f}| \\ &\geq 2|\nabla \mathrm{Re}(f)| - 3/2|\nabla\tilde{f}| \geq 13/32|\nabla \mathrm{Re}(f)|,\end{aligned}$$

we also have $|w_1| \leq Ce^{-3\rho/16}$.

Then, let $\tilde{w}_1 = e^{5\tilde{f}/4}w_1$, $\tilde{\beta}_1 = e^{5\tilde{f}/4}\beta_1$, then we have

$$\tilde{w}_0 = \tilde{w}_1 + d_{2\mathrm{Re}(f)}\tilde{\beta}_1.$$

Here $\tilde{\beta}_1$ satisfies the estimates stated above, and $\tilde{w}_1$ is $d_{2\mathrm{Re}(f)}$ closed (but may not be $L^2$ integrable).

Step 3 Now let $d_{tw,k} = e^{-k\tilde{f}/4} \circ d_{tw,0} \circ e^{k\tilde{f}/4}$. Repeating the arguments in Step 2 for 6 times, eventually we get $\alpha_6 = e^{\tilde{f}/4}w_5$ is $d_{tw,6}(= d_{2\mathrm{Re}(f)})$ closed and $L^2$ integrable. Hence we have Hodge decomposition $\alpha_6 = w_6 + d_{\mathrm{Re}(f)}\beta_6$, where $w_6$ is $\Delta_{2\mathrm{Re}(f)}$ harmonic, and $\beta_6$ satisfies the estimates stated above.

Eventually, set $w' = w_6$, $\beta := \tilde{\beta}_0 + ...\tilde{\beta}_5 + \beta_6$, we finish the proof. $\square$

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